

Codimension one foliations with Bott-Morse singularities II

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Abstract

We study codimension one foliations with singularities defined locally by Bott-Morse functions on closed oriented manifolds. We carry to this setting the classical concepts of holonomy of invariant sets and stability, and prove a stability theorem in the spirit of the local stability theorem of Reeb. This yields, among other things, a good topological understanding of the leaves one may have around a center-type component of the singular set, and also of the topology of its *basin*. The stability theorem further allows the description of the topology of the boundary of the basin and how the topology of the leaves changes when passing from inside to outside the basin. This is described via *fiberwise Milnor-Wallace surgery*. A key-point for this is to show that if the boundary of the basin of a center is non-empty, then it contains a saddle; in this case we say that the center and the saddle are *in pairing*. We then describe the possible pairings one may have in dimension three and use a construction motivated by the classical saddle-node bifurcation, that we call *foliated surgery*, that allows the reduction of certain pairings of singularities of a foliation. This is used together with our previous work on the topic to prove an extension for 3-manifolds of Reeb's recognition theorem.

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1 Introduction

Foliations with singularities on smooth manifolds appear naturally in many fields of Mathematics. For instance, given a manifold M one may: i) consider a smooth function $f : M \rightarrow \mathbb{R}$ and look at its level sets; the critical points of f yield singularities of the corresponding foliation; ii) consider a Lie group action $G \times M \rightarrow M$; the G -orbits define a foliation with singularities at the points in special orbits; iii) if M admits a Poisson structure, then this structure determines a foliation

by symplectic leaves, which is singular at the points where the rank drops.

These are just a few examples of singular foliations, there are many more. And yet, in the setting we envisage here -that of real manifolds- our knowledge of singular foliations is not so big, except in special cases. No doubt, a reason for this is the high degree of difficulty involved in the study of singular foliations: If non-singular foliation theory is already hard enough, adding singularities can turn it beyond any reasonable scope.

Thus one is naturally lead to imposing certain restrictions on the type of singular foliations one studies. Here we look at a class of codimension 1 foliations which is a natural generalization of the Morse foliations. We look at foliations which are locally defined by Bott-Morse functions. This class is large enough to be a rich family, including many interesting examples, and yet the conditions we impose do allow certain control, and one may hope to say interesting things.

The concept of foliations with Bott-Morse singularities on smooth manifolds obviously comes from the landmark work of M. Morse in his Colloquium Publication [16], as well as from R. Bott's generalization of Morse functions in [1]. This notion for foliations was introduced in our previous article [19], where we focused on the case where all singularities were transversally of center-type. The presence of saddles obviously makes the theory richer, and this is the situation we now envisage.

This type of foliations fits within the framework of *generalized foliations* introduced by H. J. Sussmann in [20], and they appear naturally in various contexts. For instance every singular Riemannian foliation in the sense of P. Molino [15] has Bott-Morse singularities of center-type; this includes the foliations given by cohomogeneity 1 actions of compact Lie groups. The foliations given by generic singular Poisson structures of corank 1 also have Bott-Morse singularities, and so do the liftings to fiber bundles of Morse foliations given on the base.

The philosophy underlying this article is the following. Suppose \mathcal{F} is a codimension 1, transversally oriented closed foliation on a closed, oriented and connected m -manifold M , and the singularities of \mathcal{F} are all of Bott-Morse type. One has that the transverse type of the foliation at each component N_j of its singular set is independent of the choice of transversal, and it is therefore determined by its Morse index. Centers correspond to the extreme cases of Morse index 0 or $m - \dim N_j$. We know from our previous article that if there are only center-type components, then there are exactly two such components, and the foliation

is given by the fibers of a smooth Bott-Morse function $f : M \rightarrow [0, 1]$ with two critical values at the points $\{0, 1\}$, which correspond to the two components of the singular set. Furthermore, in this situation the leaves are all spherical fiber bundles over each component of the singular set. Hence the topology is “well understood” in these cases.

One may thus think of the general case in the following way. Assume there is a center-type component $N_0 \subset \text{sing}(\mathcal{F})$ and look at the leaves around it, which we can describe as above; this uses a local stability theorem similar to Reeb’s theorem for compact leaves (Section 4). Look at the *basin* $\mathcal{C}(N_0, \mathcal{F})$ of N_0 (see definition inside); these are the leaves that “we understand”. Now we go to the boundary $\partial\mathcal{C}(N_0, \mathcal{F})$ of this set. By construction, if $\partial\mathcal{C}(N_0, \mathcal{F}) = \emptyset$ then $M = \mathcal{C}(N_0, \mathcal{F})$, there are no saddle singularities of \mathcal{F} in M and we are in the situation previously envisaged in [19]. Otherwise there must be a saddle component N_1 in $\partial\mathcal{C}(N_0, \mathcal{F})$ and no center-components there; in this case we say that N_0 and N_1 are *in pairing*, a key-concept for this article, inspired by [3].

If the Morse index of N_1 is not 1 nor $m - \dim N_1 - 1$, then N_1 has exactly one separatrix L_1 and if we assume that there are no saddle-connections, then the compact set $\Lambda(N_1) = N_1 \cup L_1$ is precisely the boundary $\partial\mathcal{C}(N_0, \mathcal{F})$. In sections 5 and 6 we prove a Stability Theorem (6.3) that allows us to determine the topology of $\partial\mathcal{C}(N_0, \mathcal{F})$ and of the leaves in a neighborhood of $\partial\mathcal{C}(N_0, \mathcal{F})$, but outside the basin, by comparison with the leaves inside the basin. This is possible thanks to the triviality of the holonomy of codimension one invariant subsets that we prove in Section 3.2, and the description of *distinguished neighborhoods* of saddles in Section 5. The description we give of the topology of $\partial\mathcal{C}(N_0, \mathcal{F})$ and the topology of the leaves beyond the boundary is done by means of “fiberwise Milnor-Wallace surgery” (Theorem 6.5). Of course this is inspired by ideas of R. Thom in [21].

When the Morse index of N_1 is 1 or $m - \dim N_1 - 1$ then the saddle N_1 may have one or two separatrices. If it has only one separatrix, then the discussion is exactly as above. However, if N_1 has two distinct separatrices L_1, L_2 , then $\Lambda(N_1) = N_1 \cup L_1 \cup L_2$ has two components that meet at N_1 and the topology of the leaves in $\mathcal{C}(N_0, \mathcal{F})$ only determines the topology of one of these components. We need more information in order to determine the topology of all of $\Lambda(N_1)$. As we shall see, this is actually equivalent to determining the topology of the leaves “in the other side” of the invariant set $\Lambda(N_1)$.

The hope would be that there is not “much more” beyond the boundaries of the basins of center-type components. However examples show

that this is not the case and the possibilities are infinite. Therefore we must impose additional restrictions on the foliations we consider in order to be capable of saying something. The first natural condition, that we already used above, is that the foliation has no saddle connections; we call these *Bott-Morse foliations*, see definition 2.5. We also restrict this discussion to closed foliations, which already leaves out one of the main features of foliation-theory: the presence of recurrences. Yet, for instance, if a 3-manifold admits a Heegard splitting of genus $g > 0$, then it admits closed Bott-Morse foliations with $2g$ non-isolated center components, $2g - 2$ isolated saddles and leaves of all genera $1, \dots, g$ (see Section 2.1). Moreover, if M^3 has a Heegard splitting of genus g then it has Heegard splittings of all genera $\geq g$, and therefore Bott-Morse foliations with leaves of all genera. In higher dimensions the situation is even wilder.

We thus have that the “zoo” of Bott-Morse foliations is rather big. So in the last sections of this work we restrict ourselves to dimension three, which is already rich enough. We describe several types of possible pairings one may have in 3-manifolds, completing the study done in [3] for foliations with isolated Morse singularities, and we show that if one imposes the condition of having “sufficiently more” centers than saddles, then the above mentioned list actually describes all possible pairings.

We then extend to Bott-Morse foliations a reduction technique introduced in [3], that we call *foliated surgery*, that allows us to reduce, under appropriate conditions, the number of components of the singular set of a foliation. This technique is used in [3, 4] to prove that a closed m -manifold supporting a *Morse foliation* with strictly more centers than saddles is homeomorphic to the m -sphere S^m or to a *Kuiper-Eells manifold*, *i.e.*, a closed manifold supporting a Morse function with three singular points. This generalizes the classical “sphere recognition theorem” of Reeb and an analogous result by Kuiper-Eells.

In this article we follow the same strategy, which is now somehow harder because the singular set of the foliation may have dimension > 0 and components of different dimensions. The idea is the following. Let $N \subset \text{sing}(\mathcal{F})$ be a center type component. If \mathcal{F} is not compact then $\partial\mathcal{C}(N, \mathcal{F})$, the boundary of its basin, contains some saddle singularity $N_0 \subset \text{sing}(\mathcal{F})$. Denote by $\Lambda(N_0)$ the union of N_0 and all separatrices of N_0 . Since m is 3, there are cases where $\Lambda(N_0) \setminus N_0$ is not connected. Then the “external leaves” L_e , those “beyond” the boundary $\partial\mathcal{C}(N, \mathcal{F})$, are such that $L_e \setminus (L_e \cap W)$ has two connected components, where W is a distinguished neighborhood of N . In this case our Stability

Theorem 6.3 only gives information about the connected component of $L_e \setminus (L_e \cap W)$ which is close to the basin $\mathcal{C}(N, \mathcal{F})$. The topology of the other connected component has to be controlled by some other additional information. Thence we must demand having “sufficiently more” centers than saddles: $c(\mathcal{F}) > 2s(\mathcal{F})$. With this hypothesis, the local description of \mathcal{F} in W that we get allows us to describe the topology of $L_e \cap W$. We can then use Theorem 6.5 to describe the topology of the external leaves L_e . This description allows the classification of the pairings that may possibly appear in this process, and we use foliated surgery (cf. § 7) in order to reduce the total number of singularities of \mathcal{F} but preserving the inequality $c(\mathcal{F}) > 2s(\mathcal{F})$. Repetition of this process finally shows that M can be equipped with a *compact* Bott-Morse foliation, so we use [19] to conclude (Theorem 9.1) that if M^3 is a closed oriented connected 3-manifold that admits a closed Bott-Morse foliation \mathcal{F} satisfying $c(\mathcal{F}) > 2s(\mathcal{F})$, then M is the 3-sphere, a product $S^1 \times S^2$ or a Lens space $L(p, q)$.

When the singularities are all of the same dimension, then the hypothesis $c(\mathcal{F}) > s(\mathcal{F})$ is enough to get all the information we need in order to carry on with the same strategy, and we arrive to the same conclusion (Theorem 9.1).

Simple examples (in Section 2.1) show that the above bounds ($c(\mathcal{F}) > 2s(\mathcal{F})$ in general or just $c(\mathcal{F}) > s(\mathcal{F})$ if the singular set is “pure dimensional”) are sharp in the sense that there are many closed, oriented 3-manifolds that admit closed Bott-Morse foliations with $c(\mathcal{F}) = 2s(\mathcal{F})$ singular components of mixed dimensions, and also with $c(\mathcal{F}) = s(\mathcal{F})$ singular components of the same dimension (either 0 or 1, as we please).

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2 Definitions and examples

Throughout this article M is a closed, oriented, smooth manifold (say of class C^∞ for simplicity), endowed with a riemannian metric. As usual, being closed means that M is compact and has empty boundary. Let us recall the following definition (cf. [16, 1]).

Definition 2.1. A smooth function $f : M \rightarrow \mathbb{R}$ is *Bott-Morse* if its critical points form a union of disjoint, closed, submanifolds $\bigcup_{j=1}^t N_j$ of M which are non-degenerate for f , *i.e.*, for each $p \in N_j \subset \text{sing}(\mathcal{F})$ and for each small disc Σ_p transversal to some N_j of complementary dimension, one has that the restriction $f|_{\Sigma_p}$ has an ordinary Morse singularity.

Now we have:

Let \mathcal{F} be a codimension one smooth foliation with singularities on a manifold M of dimension $m \geq 2$. We denote by $\text{sing}(\mathcal{F})$ the singular set of \mathcal{F} .

Definition 2.2 (Bott-Morse singularity). The singularities of \mathcal{F} are of *Bott-Morse type* if $\text{sing}(\mathcal{F})$ is a disjoint union of a finite number of closed connected submanifolds, $\text{sing}(\mathcal{F}) = \bigcup_{j=1}^t N_j$, each of codimension ≥ 2 , and for each $p \in N_j \subset \text{sing}(\mathcal{F})$ there exists a neighborhood V of p in M where \mathcal{F} is defined by a Bott-Morse function.

That is, if n_j is the dimension of N_j , then there is a diffeomorphism $\varphi : V \rightarrow P \times D$, where $P \subset \mathbb{R}^{n_j}$ and $D \subset \mathbb{R}^{m-n_j}$ are discs centered at the origin, such that φ takes $\mathcal{F}|_V$ into the product foliation $P \times \mathcal{G}$, where $\mathcal{G} = \mathcal{G}(N_j)$ is the foliation on D given by some function with a Morse singularity at the origin.

In other words, $\text{sing}(\mathcal{F}) \cap V = N_j \cap V$; $\varphi(N_j \cap V) = P \times \{0\} \subset P \times D \subset \mathbb{R}^{n_j} \times \mathbb{R}^{m-n_j}$, and we can find coordinates $(x, y) = (x_1, \dots, x_{n_j}, y_1, \dots, y_{m-n_j}) \in V$ such that $N_j \cap V$ is given by $\{y_1 = \dots = y_{m-n_j} = 0\}$ and $\mathcal{F}|_V$ is given by the levels of a function $J_{N_j}(x, y) = \sum_{j=1}^{m-n_j} \lambda_j y_j^2$ where $\lambda_j \in \{\pm 1\}$.

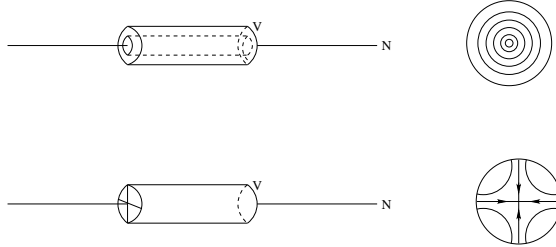


Figure 1: Center and saddle type Bott-Morse singularities

The discs $\Sigma_p = \varphi^{-1}(x(p) \times D)$ are transverse to N_j and they are transverse to \mathcal{F} outside N_j . The restriction $\mathcal{F}|_{\Sigma_p}$ is an ordinary Morse singularity and its Morse index does not depend on the point p in the component N_j , nor on the choice of the transversal slice Σ_p . We refer to $\mathcal{G}(N_j) = \mathcal{F}|_{\Sigma_p}$ as the *transverse type* of \mathcal{F} along N_j , and its *Morse index* is the Morse index of f at p .

Definition 2.3. A component $N_j \subset \text{sing}(\mathcal{F})$ is of *center-type* (or just a *center*) if the transverse type $\mathcal{G}(N_j) = \mathcal{F}|_{\Sigma_p}$ of \mathcal{F} along N_j is a center, *i.e.*, its Morse index $r = r_j$ is 0 or $m - n_j$. The component N_j is of *saddle-type* (or just a *saddle*) if its transverse type is a saddle, *i.e.*, its Morse index r is $\neq 0, m - n_j$.

In a neighborhood of a center the leaves of \mathcal{F} in the transversal Σ_p are diffeomorphic to $(m - n_j - 1)$ -spheres, where n_j is the dimension of N_j . In a neighborhood of a saddle we have leaves called *separatrices* of \mathcal{F} , which on the transversal disc are conical leaves given by expressions of the form $y_1^2 + \cdots + y_r^2 = y_{r+1}^2 + \cdots + y_{m_j}^2 \neq 0$; each such leaf contains p in its closure. Given a component $N_j \subset \text{sing}(\mathcal{F})$ whose transverse type is a saddle, a *separatrix of \mathcal{F} through N* (or simply a *separatrix of N*) is a leaf L such that its closure \overline{L} contains N_j . This means that L meets each small $(m - n_j)$ -disc Σ transversal to N_j in a separatrix of $\mathcal{F}|_{\Sigma}$.

As in the case of isolated singularities, these concepts do not depend on the choice of orientations. We denote by $\text{Cent}(\mathcal{F}) \subset \text{sing}(\mathcal{F})$ the union of center-type components in $\text{sing}(\mathcal{F})$, and by $\text{Sad}(\mathcal{F})$ the corresponding union of saddle components. We denote by $c(\mathcal{F})$ and $s(\mathcal{F})$ the number of connected components in $\text{Cent}(\mathcal{F})$ and $\text{Sad}(\mathcal{F})$ respectively.

We say that \mathcal{F} is *compact* if every leaf of \mathcal{F} is compact (and consequently $s(\mathcal{F}) = 0$). The foliation \mathcal{F} is *closed* if every leaf of \mathcal{F} is closed off $\text{sing}(\mathcal{F})$. In this case, if M is compact, then all leaves are compact except for those containing separatrices of saddles in $\text{Sad}(\mathcal{F})$: such a leaf is contained in the compact singular variety $\overline{L} \subset L \cup \text{Sad}(\mathcal{F})$, union of \overline{L} and the saddle components for which it is a separatrix. A closed foliation on a compact manifold is compact if and only if $s(\mathcal{F}) = 0$ (see [19] for details).

We say that \mathcal{F} has a *saddle-connection* if there are components $N_1 \neq N_2$ of $\text{Sad}(\mathcal{F})$ and a leaf L of \mathcal{F} which is simultaneously a separatrix of N_1 and N_2 . If a leaf L is a separatrix of \mathcal{F} through N and L meets some transversal $(m - n)$ -disc Σ in two distinct separatrices of $\mathcal{F}|_{\Sigma}$, then we say L is a *self-saddle-connection* of \mathcal{F} .

Definition 2.4 (transverse orientability). Let \mathcal{F} be a C^∞ codimension one foliation with Bott-Morse singularities on M^m , $m \geq 2$. We say that \mathcal{F} is *transversally orientable* if there exists a C^∞ vector field X on M , possibly with singularities at $\text{sing}(\mathcal{F})$, such that X is transverse to \mathcal{F} outside $\text{sing}(\mathcal{F})$.

Definition 2.5 (Bott-Morse foliation). Let \mathcal{F} be a C^∞ codimension one foliation on a differentiable manifold M^m , $m \geq 2$. We say that \mathcal{F} is a *Bott-Morse foliation* if:

- (i) The singularities of \mathcal{F} are of Bott-Morse type.
- (ii) \mathcal{F} has no saddle-connections on M (\mathcal{F} may have self-saddle-connections).
- (iii) \mathcal{F} is transversally orientable.

2.1 Examples

Example 2.6. If $\pi : M^{n+k} \rightarrow B^n$ is a C^∞ fibered bundle and the manifold B is equipped with a Bott-Morse foliation, then the inverse image of the leaves determines a Bott-Morse foliation on M . In particular, if $f : B \rightarrow \mathbb{R}$ is a Morse function with no saddle connections, then its level surfaces determine a Morse foliation on B which lifts to a Bott-Morse foliation on M .

Example 2.7. A *Poisson structure* on a smooth manifold M consists of vector bundle morphism $\psi : T^*M \rightarrow TM$ satisfying an integrability condition, whose rank at each point is called the rank of the Poisson structure. If the rank is constant then the integrability condition implies one has a foliation on M , of dimension equal to the rank; the tangent space of the foliation at each point $x \in M$ is the image of $\psi(T_x^*M)$ in T_xM . If the rank is not constant then one has a singular foliation with singularities at the points where the rank drops. The Dolbeault-Weinstein theorem implies that at such points the transversal structure plays a key-role, and generically the transverse structure is given by a Morse function.

Example 2.8. Every codimension 1 singular Riemannian foliation in the sense of P. Molino (see for instance the last chapter of his book [15]; see also [10] for more on the subject), is Bott-Morse with only center-type components. This includes the foliations defined by cohomogeneity 1 isometric actions of Lie groups on smooth manifolds with special orbits.

Let us now give some examples of Bott-Morse foliations on 3-manifolds which are important in the sequel.

Example 2.9. Every closed oriented 3-manifold can be expressed as a union $M^3 = L(g) \cup L(g)'$ where $L(g), L(g)'$ are solid handlebodies of genus $g \geq 0$, glued along their boundary S_g . These are called *Heegard splittings* (or *decompositions*) of M , and g is the genus of the corresponding decomposition. The sphere is the only 3-manifold admitting such a splitting with genus 0. If M has a splitting of genus g then it has splittings of all genera $\geq g$.

Given a Heegard splitting $M^3 = L(g) \cup L(g)'$ one can take a product neighborhood $S_g \times [-\epsilon, \epsilon]$ of S_g and foliate it by surfaces of genus g , parallel to the boundary, with S_g corresponding to $S_g \times \{0\}$. On the level $S_g \times \{\epsilon\}$ take circles C_1, \dots, C_{g-1} separating this surface into g components, each of genus 1, and deform each of these circles to a point (see Figure 2). We get inside $L(g)$ a singular surface S with $g - 1$ saddle-points, which splits $L(g)$ into $(g + 1)$ -components: an “outer” one, diffeomorphic to $\partial S_g \times [0, \epsilon)$; and g “inner” components, each diffeomorphic to an open solid torus T_j , $j = 1, \dots, g$. We can now foliate each T_j in the usual way, by copies of $S^1 \times S^1$, having in each a circle N_j as singular set, all of center-type. We can do the same construction on the other handle-body $L(g)'$ and get a foliation on M^3 with Bott-Morse singularities. Notice this foliation has the surface S as a separatrix through each singular point, so \mathcal{F} has saddle-connections, but it is easy to change the construction slightly to get a Bott-Morse foliation. For instance, in the above construction, deform only the first circle C_1 to a point, getting a surface S_1 with one saddle singularity and bounding two “inner” components, one, say T_1 , of genus 1 and another $L(g - 1)$ of genus $g - 1$. Foliate the “exterior” of S_1 as before, by surfaces of genus g , and foliate the torus T_1 as before, with a center-type singular set. Foliate also a neighborhood in $L(g - 1)$ of its boundary by surfaces of genus $g - 1$. Now choose one of these surfaces of genus $g - 1$ and choose on it a circle that separates a handle from the others and repeat the previous construction. One gets a new separatrix, a new torus foliated by concentric tori, and an open solid region of genus $g - 2$, etc. We get finally a foliation on $L(g)$ with $g - 1$ separatrices, each with an isolated saddle singularity, g foliated tori, each with a non-isolated center-type singularity, and leaves of all genera $g - 1, g - 2, \dots, 1$ filling out $L(g)$; and similarly for $L(g)'$.

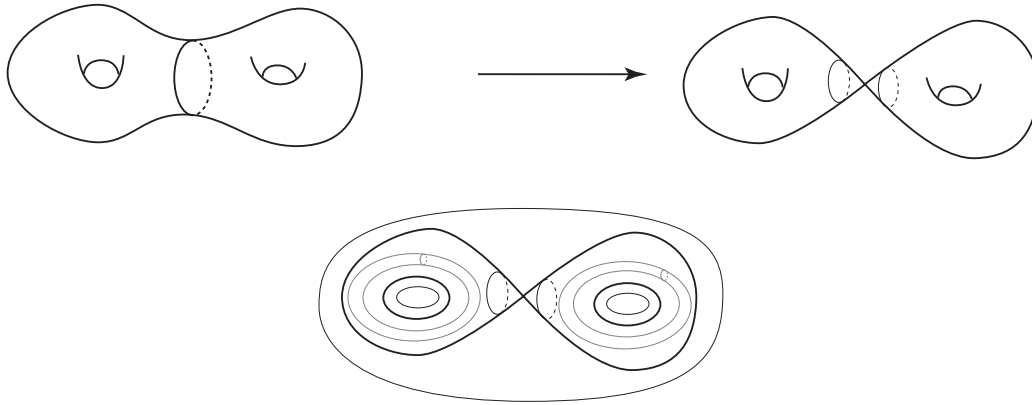


Figure 2: An example of the construction of Example 2.9 for genus $g = 2$.

Example 2.10. The sphere S^3 admits a Bott-Morse foliation with singular set consisting of four isolated centers and a non-isolated saddle as in Figure 3; the fourth center is at infinity. We can also foliate S^3 with Bott-Morse singularities consisting of a non-isolated saddle, two isolated centers and a non-isolated center (Figure 4). In both constructions an isolated center is *at infinity* with respect to the saddle.

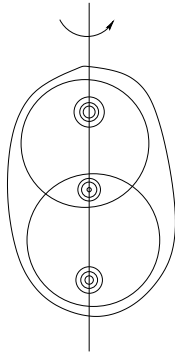


Figure 3: A foliation of S^3 by four isolated centers and a non-isolated saddle.

Example 2.11. A closed three-manifold M admits a transversally oriented non-singular compact foliation \mathcal{F}_0 if and only if it fibers over the

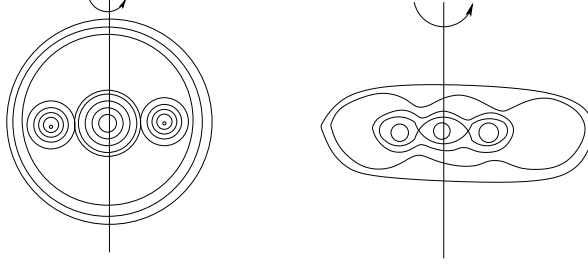


Figure 4: A foliation of S^3 by two isolated centers, a non-isolated saddle and a non-isolated center.

circle S^1 . Every such manifold can be equipped with a closed Bott-Morse foliation \mathcal{F} such that $c(\mathcal{F}) = 2s(\mathcal{F}) = 2k$, where $k \in \mathbb{N}$ is any given natural number. In fact, given any codimension 1 foliation on a 3-manifold M , one can replace a flow-box where the foliation is regular, by a box where the new foliation has two isolated centers and one non-isolated saddle; this is depicted, in reverse order, in Figure 6. This process can be repeated as many times as we want. If the original foliation was closed, so is the new one. Thus the equality $c(\mathcal{F}) = 2s(\mathcal{F})$ is not restrictive (cf. Theorem 9.1): every closed, oriented 3-manifold M admits Bott-Morse foliations with $c(\mathcal{F}) = 2s(\mathcal{F}) = 2k$ for each integer $k \geq 1$; and if M fibers over S^1 then the foliation can be chosen to be closed.

Notice that in the previous construction the saddles and the centers have different dimensions.

Similarly, given a closed manifold M of dimension $m \geq 2$, equipped with a non-singular compact foliation, we can modify the foliation as follows. Choose a flow box region $R \subset M$ and replace the foliation in R by a foliation with an isolated center and an isolated saddle of type $x_m^2 - \sum_{j=1}^{m-1} x_j^2 = 0$, obtained by rotating the Figure 5 with respect to an axe that passes through the center and the saddle.

This produces a closed Bott-Morse foliation with k isolated centers and k isolated saddles. Now, we consider a three-manifold M equipped with a non-singular compact foliation \mathcal{F} . Let L_0 be a leaf of \mathcal{F} and $\gamma_0 \subset L_0$ a C^∞ closed path. The holonomy of γ_0 is trivial and we can take a section Σ transverse to γ_0 , diffeomorphic to the square $Q = (-1, 1) \times (-1, 1)$,

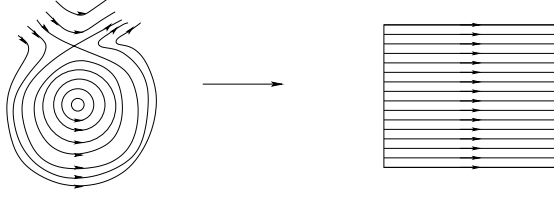


Figure 5:

with $\Sigma \cap \gamma_0 = \{p_0\}$, and a neighborhood U of γ_0 diffeomorphic to the solid torus $S^1 \times Q$. In this neighborhood we can replace \mathcal{F} by the foliation obtained as the product by S^1 of the two-dimensional foliation on the left in Figure 5. Repetition of this process yields closed Bott-Morse foliations on M with k non-isolated centers and k non-isolated saddles as singular set (cf. Theorem 9.1).

Example 2.12. On the other hand, foliations with Bott-Morse singularities satisfying the inequality $c(\mathcal{F}) > 2s(\mathcal{F})$ can be easily obtained in any of the manifolds appearing in Theorem 9.1. In fact, since these manifolds can be obtained by gluing two solid tori along their boundary, one can foliate each torus by concentric tori and get a foliation with two center components and no saddles.

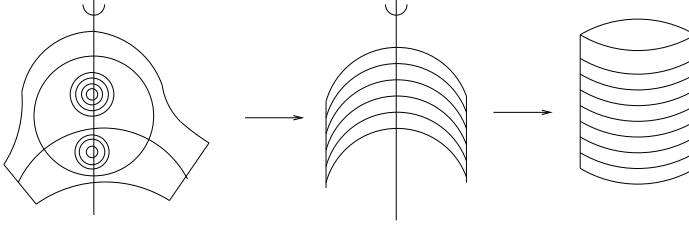


Figure 6: Reversing the arrows we get two isolated centers and a non-isolated saddle from a flow box.

Example 2.13. We consider a closed 3-manifold M with a *nonsingular* codimension one foliation \mathcal{F}_o with a Reeb component $R \subset M$. In the solid torus region R we replace \mathcal{F}_o by a singular compact Bott-Morse foliation by concentric tori. The resulting foliation \mathcal{F} on M can be assumed smooth. It is a Bott-Morse foliation with only center-type

components but it is not closed, and the manifold M may not be as in Theorem 9.1.

Example 2.14. Let T^2 be a closed surface equipped with a foliation \mathcal{F}_1 defined by a Morse function with exactly three singularities, two of center-type and one of saddle-type, and let M^3 be an S^1 -bundle over T^2 . Then, as in the first example above, \mathcal{F}_1 lifts to a Bott-Morse foliation \mathcal{F} on M with singular set consisting of three embedded circles, two of them being (non-isolated) centers and the other a saddle. This foliation satisfies the inequality on Theorem 9.1 and M need not be as claimed in that theorem. Nevertheless, according to Eells and Kuiper ([5], [6]) the only possibility for T^2 is $T^2 = \mathbb{R}P(2)$ and the resulting manifold M is non-orientable.

3 Holonomy of invariant subsets

In this section we extend to Bott-Morse foliations the classical notion of holonomy of leaves, that we recall below.

3.1 Holonomy of a leaf

This notion is originally found in the work of Ehresmann and Shih [7] and was further developed in the subsequent work of Reeb [17]. Let \mathcal{F} be a codimension k foliation on a manifold M of dimension $m = k + l$. A *distinguished* neighborhood for \mathcal{F} in M is an open subset $U \subset M$ with a coordinate chart $\varphi: U \rightarrow \varphi(U) \subset \mathbb{R}^m$ such that $\varphi(U) = D_1^l \times D_2^k$ is the product of disks in $\mathbb{R}^m = \mathbb{R}^l \times \mathbb{R}^k$ and the leaves of the restriction $\mathcal{F}|_U$ (i.e., the plaques of \mathcal{F} in U) are of the form $\varphi^{-1}(D_1^l \times \{y\})$, $y \in D_2^k$. If $V \subset U$ is another distinguished open set, we say that V is *uniform* in U if every plaque of \mathcal{F} in U meets at most one plaque of \mathcal{F} in V . This means that the natural map on leaf spaces $V/\mathcal{F}|_V \rightarrow U/\mathcal{F}|_U$ is injective. In codimension one every distinguished open set V contained in another distinguished open set U is always uniform. In general given a finite collection of distinguished open sets U_1, \dots, U_r for \mathcal{F} in M , every point in the intersection $U_1 \cap \dots \cap U_r$ has a fundamental system of distinguished open sets which are uniform with respect to each U_j (cf. [8], Lemma 1.2 page 71).

A locally finite open covering $\mathcal{U} = \{U_j\}_{j \in J}$ of M is *regular* for \mathcal{F} if: (1) each open set U_j is distinguished for \mathcal{F} ; and (2) any two or three open subsets of \mathcal{U} having a connected intersection are uniform with

respect to a same distinguished open subset for \mathcal{F} . In particular, (3) each plaque of an open subset in \mathcal{U} meets at most one plaque of another open set in \mathcal{U} . Every open cover of M can be refined into a regular cover ([8] Proposition 1.6 page 73).

A *chain* of open subsets of \mathcal{U} is a finite collection $\mathcal{C} = \{U_1, \dots, U_r\}$ of open subset in \mathcal{U} such that two consecutive elements have non-empty intersection. The chain \mathcal{C} is *closed* if $U_r = U_1$.

Let now $\mathcal{U} = \{U_j\}_{j \in J}$ be a regular covering of M with respect to \mathcal{F} and for each index $j \in J$ denote by Σ_j the leaf space of $\mathcal{F}|_{U_j}$ with projection $\pi_j: U_j \rightarrow \Sigma_j$. The foliation charts $\varphi_j: U_j \rightarrow \mathbb{R}^m = \mathbb{R}^l \times \mathbb{R}^k$ allow us to identify the leaves spaces Σ_j with sections transverse to \mathcal{F} in the U_j . By the uniformity of the open sets in \mathcal{U} , if $U_i \cap U_j \neq \emptyset$ then there is a local diffeomorphism $h_{ij}: \Sigma_i \rightarrow \Sigma_j$ such that $\pi_j = h_{ij} \circ \pi_i$ on $U_i \cap U_j$; we also have $h_{ji} = h_{ij}^{-1}$ and on each non-empty intersection $\pi_i(U_i \cap U_j \cap U_u)$ we have $h_{uj} \circ h_{ji} = h_{ui}$. The collection $\mathcal{H}(\mathcal{F})$ of local diffeomorphisms h_{ij} defines the *holonomy pseudo-group* of \mathcal{F} with respect to the regular covering \mathcal{U} . By the above properties of regular coverings this holonomy pseudogroup is intrinsically defined by the foliation \mathcal{F} and its localization to a leaf L of \mathcal{F} gives the *holonomy group* of the leaf L .

The following result comes from the proof of the Complete Stability Theorem of Reeb (cf. [8]):

Proposition 3.1. *Let \mathcal{F} be a transversely oriented, codimension one, nonsingular closed foliation on a connected manifold T , not necessarily compact.*

- (i) *Let L be a compact leaf of \mathcal{F} and let L_n be a sequence of compact leaves of \mathcal{F} accumulating to L . Then given a neighborhood W of L in T one has $L_n \subset W$ for all n sufficiently large.*
- (ii) *Assume that \mathcal{F} has a compact leaf with trivial holonomy and let $\Omega(\mathcal{F})$ be the set of compact leaves $L \in \mathcal{F}$ with trivial holonomy. Then $\Omega(\mathcal{F})$ is open in T and $\partial\Omega(\mathcal{F})$ contains no compact leaf. Indeed, a compact leaf which is a limit of compact leaves with trivial holonomy also has trivial holonomy.*
- (iii) *Let L be a compact leaf with finite holonomy group. Then the holonomy of L is trivial and there is a fundamental system of invariant neighborhoods W of L such that $\mathcal{F}|_W$ is equivalent to the product foliation on $L \times (-1, 1)$ with leaves $L \times \{t\}$.*

3.2 Holonomy of a component of the singular set

We consider again a Bott-Morse foliation \mathcal{F} . We recall the notion, introduced in [19], of *holonomy* of a component N of the singular set of \mathcal{F} of dimension $n \geq 0$. Consider a finite open cover $\mathcal{U} = \{U_1, \dots, U_\ell, U_{\ell+1}\}$ of N by open subsets $U_j \subset M$ with $U_{\ell+1} = U_1$ and charts $\varphi_j: U_j \rightarrow \varphi_j(U_j) \subset \mathbb{R}^m$ with the following properties:

(1) Each $\varphi_j: U_j \rightarrow \varphi_j(U_j) \subset \mathbb{R}^m$ defines a local product trivialization of \mathcal{F} ; $\varphi_j(U_j \cap N)$ is an n -disc D_j and $\varphi_j(U_j)$ is the product of D_j by an $m - n$ disc.

(2) $U_j \cap U_{j+1} \neq \emptyset, \forall j = 1, \dots, \ell$.

(3) If $U_i \cap U_j \neq \emptyset$ then there exists an open subset $U_{ij} \subset M$ containing $U_i \cup U_j$ and a chart $\varphi_{ij}: U_{ij} \rightarrow \varphi_{ij}(U_{ij}) \subset \mathbb{R}^m$ of M , such that φ_{ij} defines a product structure for \mathcal{F} in U_{ij} and $U_{ij} \cap N \supset [(U_i \cup U_j) \cap N] \neq \emptyset$. In each U_j we choose a transverse disc Σ_j , $\Sigma_j \cap N = \{q_j\}$ such that $\Sigma_{j+1} \subset U_j \cap U_{j+1}$ if $j \in \{1, \dots, \ell\}$.

In each U_j the foliation is given by a smooth function $F_j: U_j \rightarrow \mathbb{R}$ which is the natural trivial extension of its restriction to any of the transverse discs Σ_j or Σ_{j+1} . There is a C^∞ local diffeomorphism $\varphi_j: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ such that $F_{j+1}|_{\Sigma_{j+1}} = \varphi_j \circ F_j|_{\Sigma_{j+1}}$. This implies that $F_{j+1} = \varphi_j \circ F_j$ in $U_j \cap U_{j+1}$. Notice that, as in the classical case of non-singular foliations (see [2] chapter II or [8] Definition 1.5 page 72), by condition (3), if $U_i \cap U_k \neq \emptyset$, then the existence of the maps φ_{ij} grants that every plaque \mathcal{F} in $U_i \setminus N$ intersects at most one plaque of $U_k \setminus N$. The *holonomy map associated to N* is the local diffeomorphism $\varphi: (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$ defined by the composition $\varphi = \varphi_\ell \circ \dots \circ \varphi_1$. This map is well-defined up to conjugacy in $\text{Diff}(\mathbb{R}, 0)$.

3.3 Holonomy of invariant subsets of codimension one

We now extend the notion of holonomy to connected invariant subsets Λ of codimension one of a Bott-Morse foliation \mathcal{F} . If Λ is a compact leaf, this only means its holonomy as a leaf of the restriction \mathcal{F}_0 of \mathcal{F} to $M \setminus \text{sing } \mathcal{F}$, and if Λ is a component of $\text{sing}(\mathcal{F})$, then its holonomy was introduced above. The new case is when Λ is the union of a saddle-type singular component (of arbitrary dimension ≥ 0) with some of its separatrices.

Let us assume first that $N_0 = \{p\}$ is an isolated saddle and $\Lambda = N_0 \cup \tau$, where τ is union of separatrices; we follow [3]. Notice that in a small neighborhood of p , τ can consist of one or two components τ_1 and

τ_2 , and that this can only happen if p is a 1 or $m - 1$ saddle. In this case Λ locally divides the manifold M into three connected components. One of them, say R_3 , is the union of (regular) leaves which are hyperboloids of one sheet, and the others, say R_1 and R_2 , are union of one connected components of hyperboloids of two sheets, as depicted in Figure 7. Let $\gamma: [0, 1] \rightarrow \Lambda$ be a piecewise smooth path on Λ which passes through the singularity p , going from τ_1 to τ_2 . Fix a neighborhood U of $p \in \text{sing } \mathcal{F}$ where \mathcal{F} is given by a Morse function f with a unique singularity at p . Using the level sets of f , the holonomy along γ can be defined in the usual way on R_3 , by lifting paths to the leaves. Let us extend this map to the other regions R_1 and R_2 . Let T_0 and T_1 be local transversals to \mathcal{F} at $\gamma(0)$ and $\gamma(1)$ respectively. The *holonomy* along γ is the map which carries $t \in T_0$ to $f^{-1}(f(t)) \cap T_1 \in T_1$. This holonomy map is well-defined even if γ is not contained in $\{p\} \cup \tau_1$.

The extension of this concept to the case when the isolated saddle has Morse index different from 1 and $m - 1$ is just as in the case of the region R_3 above, so we leave it to the reader.

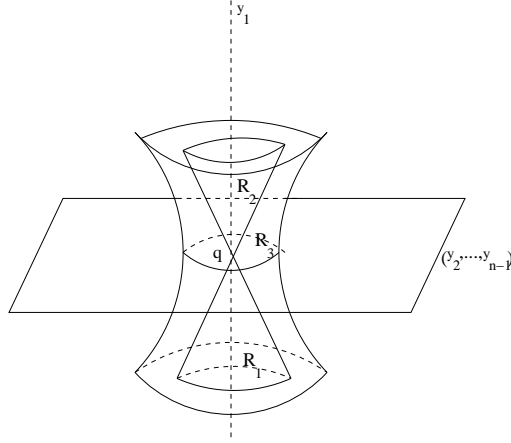


Figure 7: Holonomy of an isolated saddle

Now we consider the case when N_0 is non-isolated of dimension $n_0 \geq 1$. Let $\gamma: [0, 1] \rightarrow \Lambda$ be a path for which we wish to define the holonomy map. We can assume the image of γ is contained in a neighborhood U of some point $p \in N_0$, which is diffeomorphic to a product $D_0 \times V$, where $D_0 \subset \mathbb{R}^{n_0}$ is the unit disc centered at the origin and $V \subset \mathbb{R}^{m-n_0}$ gives the transverse model of \mathcal{F} along N_0 . In other

words, in U the foliation is given by a Bott-Morse function $f: U \rightarrow \mathbb{R}$ of product type: there are local coordinates $(x, y) \in U \cong D_0 \times V$ such that $f(x, y) = g(y)$ where $g(y)$ is a Morse function which describes the transverse type of \mathcal{F} along N_0 . Then we can introduce the holonomy map h associated to the path $\gamma \subset U$ using exactly the same construction as above, by setting $h: T_0 \rightarrow T_1$ to be the map which carries $t \in T_0$ into $f^{-1}(f(t)) \cap T_1 \in T_1$.

The following result is used in the sequel. When the invariant set is a compact leaf, this is the classical Reeb local stability theorem ([8], [2]), extended in [19] to the case of centers.

Theorem 3.2. *Let $\Lambda \subset M$ be a compact leaf, a component of the singular set or the union of a saddle component $N \subset \text{sing}(\mathcal{F})$ with some of its separatrices. Then:*

1. *The holonomy group of Λ is trivial.*
2. *There is a fundamental system of invariant neighborhoods W_α of Λ in M such that on each W_α the foliation $\mathcal{F}|_{W_\alpha}$ is given by a Bott-Morse function $f_\alpha: W_\alpha \rightarrow \mathbb{R}$.*

Proof. In all the cases, since \mathcal{F} is closed, any leaf close enough to Λ but not contained in Λ , is compact. Therefore a holonomy map h corresponds to a local diffeomorphism $h \in \text{Diff}^\infty(\mathbb{R}, 0)$ such that every orbit of h is finite. Since h is orientation preserving this implies (as in [2] Lemma 5 page 72) that $h = \text{Id}$, proving the first statement. Assume first that $\Lambda = L$ is a compact leaf. Since it has trivial holonomy, the classical local stability theorem implies that L admits a fundamental system of invariant neighborhoods W where \mathcal{F} is a trivial product foliation, and in particular given by a submersion. Assume now that Λ is a center type component in $\text{sing}(\mathcal{F})$. Theorem B (actually, Lemma 2.5) in [19] shows that there is a fundamental system of invariant neighborhoods of Λ where \mathcal{F} is equivalent to a fibre bundle over Λ . Finally, assume that Λ is the union of a saddle $N \subset \text{sing}(\mathcal{F})$ and some of its separatrices. Because the holonomy of Λ is trivial, *a fortiori* also the holonomy of $N \subset \text{sing}(\mathcal{F})$ is trivial and we apply Lemma 2.5 in [19] to obtain a Bott-Morse function f_0 which defines the foliation in a small neighborhood $U(N)$ of N . Again because the holonomy of Λ is trivial, classical holonomy extension arguments allow us to extend f_0 as a first integral to \mathcal{F} in a neighborhood W of Λ , constructed as the saturated of $U(N)$. \square

4 Basins of Centers

In this section we look at the topology of the foliation near a center-type component.

4.1 Stability of foliations with center-type singularities

We recall first the main results of [19] that we use in the sequel.

Definition 4.1. Let \mathcal{F} be a possibly singular foliation on M . A subset $B \subset M$, invariant by \mathcal{F} , is *stable* (for \mathcal{F}) if for any given neighborhood W of B in M there exists a neighborhood $W' \subset W$ of B such that every leaf of \mathcal{F} intersecting W' is contained in W .

The following is essentially Proposition 2.7 in [19]:

Proposition 4.2. *Let \mathcal{F} be a Bott-Morse foliation on M . Given a compact component $N \subset \text{sing}(\mathcal{F})$ we have:*

1. *If N is of center type and it is a limit of compact leaves then N is stable.*
2. *If \mathcal{F} is compact in a neighborhood of N , then N is stable of center type with trivial holonomy.*
3. *If N is of center type and the holonomy group of N is finite then N is stable and the nearby leaves are all compact.*

One has the following the Local Stability Theorem in [19] (Theorem B):

Theorem 4.3. *Let \mathcal{F} be as before, a closed Bott-Morse foliation on M^m , and let $N^n \subset \text{sing}(\mathcal{F})$ be transversely a center. Then N is stable and there is a fundamental system of invariant compact neighborhoods $\{W_\nu\}$ of N such that every leaf in W_ν is compact, with trivial holonomy and diffeomorphic to the unit sphere normal bundle of N in M . Hence every such leaf is an $(m - n - 1)$ -sphere bundle over N .*

And one also has the corresponding Complete Stability Theorem (Theorem A in [19]):

Theorem 4.4. *Let \mathcal{F} be a smooth foliation with Bott-Morse singularities on a closed oriented manifold M of dimension $m \geq 3$ having*

only center type components in $\text{sing}(\mathcal{F})$. Assume that \mathcal{F} has some compact leaf L_o with finite fundamental group, or there is a codimension ≥ 3 component N with finite fundamental group. Then all leaves of \mathcal{F} are compact, stable, with finite fundamental group. If, moreover, \mathcal{F} is transversally orientable, then $\text{sing}(\mathcal{F})$ has exactly two components and there is a differentiable Bott-Morse function $f: M \rightarrow [0, 1]$ whose critical values are $\{0, 1\}$ and such that $f|_{M \setminus \text{sing}(\mathcal{F})}: M \setminus \text{sing}(\mathcal{F}) \rightarrow (0, 1)$ is a fiber bundle with fibers the leaves of \mathcal{F} .

The proof of this theorem actually shows that every compact transversely oriented foliation with non-empty singular set, all of Bott-Morse type, has exactly two components in its singular set and is given by a Bott-Morse function $f: M \rightarrow [0, 1]$ as is in the statement. This theorem and its proof lead to the following generalization of [11, Theorem 1.5], which provides a complete topological characterization of the manifolds and foliations having only center-type singularities:

Theorem 4.5. *Let \mathcal{F} be a transversally oriented, compact foliation with Bott-Morse singularities on a closed, oriented, connected manifold M^m , $m \geq 3$, with non-empty singular set $\text{sing}(\mathcal{F})$. Let L be a leaf of \mathcal{F} . Then $\text{sing}(\mathcal{F})$ has two connected components N_1, N_2 , both of center type, and one has:*

- (i) $M \setminus (N_1 \cup N_2)$ is diffeomorphic to the cylinder $L \times (0, 1)$.
- (ii) L is a sphere fiber bundle over both manifolds N_1, N_2 and M is diffeomorphic to the union of the corresponding disc bundles over N_1, N_2 , glued together along their common boundary L by some diffeomorphism $L \rightarrow L$.
- (iii) In fact one has a double-fibration

$$N_1 \xleftarrow{\pi_1} L \xrightarrow{\pi_2} N_2 ,$$

and M is homeomorphic to the corresponding mapping cylinder, i.e., to the quotient space of $(L \times [0, 1]) \cup (N_1 \cup N_2)$ by the identifications $(x, 0) \sim \pi_1(x)$ and $(x, 1) \sim \pi_2(x)$.

This yields to an explicit description of this type of foliations on manifolds of dimensions 3 (and 4), that will be used later in this article:

Theorem 4.6. *Let M be a closed oriented connected 3-manifold equipped with a transversely oriented compact foliation \mathcal{F} with Bott-Morse singularities. Then either $\text{sing}(\mathcal{F})$ consists of two points, the leaves are*

2-spheres and M is S^3 , or $\text{sing}(\mathcal{F})$ consists of two circles, the leaves are tori and M is homeomorphic to the 3-sphere S^3 , a Lens space or a product $S^2 \times S^1$.

The 3-manifolds that appear in this theorem are exactly those admitting a Heegard splitting of genus 1 (see for instance [9]).

4.2 Topology of the basin of a center

The previous results, particularly Theorem 4.3, motivate the following definition, which is one of the main concepts in this article:

Definition 4.7 (Basin of a center). We define the set $\mathcal{C}(\mathcal{F}) \subset M$ as the union of all the centers of \mathcal{F} and all the compact leaves $L \in \mathcal{F}$ which bound a compact invariant region $R(L, N)$, neighborhood of some center type component $N \subset \text{sing}(\mathcal{F})$, of dimension $n \geq 0$, with the following properties:

- 1) The region $R(L, N)$ is equivalent to a fibre bundle with fibre the closed disc \overline{D}^{m-n} over N , the fibers being transversal to the leaves of \mathcal{F} in $R(L, N)$.
- 2) Each leaf $L \subset R(L, N)$ is a fibre bundle over N with fiber the sphere S^{m-n-1} .

Given a center type component $N \subset \text{sing}(\mathcal{F})$ the *basin* of N , denoted $\mathcal{C}(N) = \mathcal{C}(N, \mathcal{F})$, is the connected component of $\mathcal{C}(\mathcal{F})$ that contains the center N .

We have:

Theorem 4.8. *Let \mathcal{F} be a closed Bott-Morse foliation on M and N_1, N_2 center type components of the singular set of \mathcal{F} . Then the basins $\mathcal{C}(N_i, \mathcal{F})$ are open sets in M , either disjoint or identical, i.e., $\mathcal{C}(N_1, \mathcal{F}) \cap \mathcal{C}(N_2, \mathcal{F}) = \emptyset$ or $\mathcal{C}(N_1, \mathcal{F}) = \mathcal{C}(N_2, \mathcal{F})$, and we have:*

1. *If the boundary $\partial\mathcal{C}(N_1, \mathcal{F})$ is empty, then $\mathcal{C}(N_1, \mathcal{F}) = M$. In this case the singular set of \mathcal{F} consists of exactly two center type components, say N_1, N_2 (there are no saddles), the foliation is compact, given by a Bott-Morse function $f : M \rightarrow [0, 1] \subset \mathbb{R}$, and each leaf is diffeomorphic to the boundary of a tubular neighborhood of both manifolds N_1, N_2 , so it is a fibre bundle over both N_1 and N_2 with fibre a sphere of the appropriate dimension.*

2. If $\partial\mathcal{C}(N_1, \mathcal{F}) \neq \emptyset$, then $\mathcal{C}(N_1, \mathcal{F})$ is diffeomorphic to the total space of the normal bundle of N_1 in M , and there is exactly one saddle component N_0 of $\text{sing}(\mathcal{F})$ in $\partial\mathcal{C}(N, \mathcal{F})$. In this case $\partial\mathcal{C}(N, \mathcal{F})$ is the union of N_0 and separatrices of N_0 .
3. If $\mathcal{C}(N_1, \mathcal{F}) \neq \mathcal{C}(N_2, \mathcal{F})$ and $\partial\mathcal{C}(N_1, \mathcal{F}) = \partial\mathcal{C}(N_2, \mathcal{F})$, then $M = \overline{\mathcal{C}(N_1, \mathcal{F}) \cup \mathcal{C}(N_2, \mathcal{F})}$.
4. If $\partial\mathcal{C}(N_1, \mathcal{F}) \neq \partial\mathcal{C}(N_2, \mathcal{F})$ and $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) \neq \emptyset$, then there is a saddle component $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ of Morse index 1 or $m - n_o - 1$, where n_o is the dimension of N_0 .

Proof. First notice that Theorems 4.3 and 3.2 imply that the sets $\mathcal{C}(N_i, \mathcal{F})$ are non-empty. That they are open sets in M , and either they are disjoint or identical is immediate from the definition of the basin and Reeb's Local Stability Theorem (see Theorem 3.2).

To prove the statement (1), recall that $\mathcal{C}(N_1, \mathcal{F})$ is an open subset of M ; if it has empty boundary, then it is also closed and therefore $\mathcal{C}(N_1, \mathcal{F}) = M$, by connecteness. Hence the foliation is compact and there are no saddles. Then the rest of statement (1) follows immediately from [19].

Let us prove statement (2). If $\partial\mathcal{C}(N_1, \mathcal{F}) \neq \emptyset$, we claim that $\mathcal{C}(N_1, \mathcal{F})$ is the union of N_1 and all the compact leaves that bound a tubular neighborhood of N_1 . This obviously implies that $\mathcal{C}(N_1, \mathcal{F})$ is diffeomorphic to the normal bundle of N_1 in M .

Claim 4.9. $\mathcal{C}(N_1, \mathcal{F})$ that is the union of N_1 and all the compact leaves that bound a tubular neighborhood of N_1 .

Proof of the claim. For a leaf $L \subset \partial\mathcal{C}(N, \mathcal{F})$ close enough to N_1 it is clear that L bounds a tubular neighborhood $R(L)$ of N_1 in M . Applying the local stability theorem of Reeb to L this same property holds for any leaf L_1 close enough to L . Thus, by connecteness of $\partial\mathcal{C}(N, \mathcal{F})$ it follows that any leaf in $\partial\mathcal{C}(N, \mathcal{F})$ bounds a tubular neighborhood of N_1 . The same argumentation actually shows that $\partial\mathcal{C}(N, \mathcal{F})$ is the union of N_1 and all compact leaves that bound a tubular neighborhood of N_1 with projection transverse to the leaves and fiber a disc. \square

The above arguments prove the first claim in (2). Let us prove now that there is exactly one saddle component of $\text{sing}(\mathcal{F})$ in $\partial\mathcal{C}(N, \mathcal{F})$. First we show there is no compact leaf in $\partial\mathcal{C}(N_1, \mathcal{F})$. If $L \subset \partial\mathcal{C}(N_1, \mathcal{F})$ is compact and accumulated by a sequence of leaves $L_\nu \subset \mathcal{C}(N_1, \mathcal{F})$, $\nu \in \mathbb{N}$,

then given W as in Theorem 3.2 we have $L_\nu \subset W$ for $\nu \gg 1$ and since \mathcal{F} is of product type in W we have $W \subset \mathcal{C}(N_1, \mathcal{F})$ so that $L \not\subset \partial\mathcal{C}(N_1, \mathcal{F})$, a contradiction. Therefore every leaf $L \subset \partial\mathcal{C}(N_1, \mathcal{F})$ is separatrix of some saddle component. By definition, if a center component \tilde{N} is accumulated by leaves in $\mathcal{C}(N_1, \mathcal{F})$ then \tilde{N} is contained in $\mathcal{C}(N_1, \mathcal{F})$. Hence there are no centers in $\partial\mathcal{C}(N, \mathcal{F})$. On the other hand, given a saddle component $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F})$ it is clear that some separatrix of N_0 is contained in $\partial\mathcal{C}(N_1, \mathcal{F})$. Let us prove that there is exactly one saddle component in $\partial\mathcal{C}(N_1, \mathcal{F})$. If N, N_0 are different saddles in $\partial\mathcal{C}((N_1, \mathcal{F}))$ then there is a sequence of compact leaves $L_\nu \subset \mathcal{C}((N_1, \mathcal{F}))$, $\nu \in \mathbb{N}$, which accumulate both N_1 and N_2 as $\nu \rightarrow \infty$. Hence there exist separatrices \mathcal{L} and \mathcal{L}_0 of N and N_0 , respectively, which are accumulated by the L_ν . Notice that the sets $\Lambda = \mathcal{L} \cup N$ and $\Lambda_0 = \mathcal{L}_0 \cup N_0$ are both compact and invariant, so by Theorem 3.2 they have trivial holonomy and each of these sets has a fundamental system W_ν, W_{ν_0} of invariant neighborhoods. If $\mathcal{L} \neq \mathcal{L}_0$ then we have $\Lambda \cap \Lambda_0 = \emptyset$ and therefore $W_\nu \cap W_{\nu_0} = \emptyset$ for W_ν, W_{ν_0} small enough. On the other hand we have $L_\nu \subset W_\nu$ and $L_\nu \subset W_{\nu_0}$ for all $\nu, \nu_0 \gg 1$, a contradiction, since by hypothesis there are no saddle connections. This proves (2).

For (3) we notice that if $\mathcal{C}(N_1, \mathcal{F}) \neq \mathcal{C}(N_2, \mathcal{F})$ and $\partial\mathcal{C}(N_1, \mathcal{F}) = \partial\mathcal{C}(N_2, \mathcal{F})$, then $\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}$ is open and obviously closed in M , so the statement follows by connectedness.

Finally we prove (4). Suppose that $\partial\mathcal{C}(N_1, \mathcal{F}) \neq \partial\mathcal{C}(N_2, \mathcal{F})$ and $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) \neq \emptyset$. By (ii) there is a single saddle component $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$. If the transverse Morse index of N_0 is different from 1 and $m - n_0 - 1$, then in suitable local coordinates $(x_1, \dots, x_{n_0}, y_1, \dots, y_{m-n_0}) \in M$ we have $N_0 = \{y_1 = \dots = y_{m-n_0} = 0\}$ and the union $\Lambda(N_0)$ of N_0 and the local separatrix through N_0 is given by $y_1^2 + \dots + y_r^2 = y_{r+1}^2 + \dots + y_{m-n_0}^2$ where $r \notin \{1, m - n_0 - 1\}$. Hence the local separatrix is connected. This implies N_0 has only one separatrix and therefore, by (ii), $\partial\mathcal{C}(N_1, \mathcal{F}) = \partial\mathcal{C}(N_2, \mathcal{F})$, which is a contradiction. Hence the transverse Morse index of N_0 must be as stated in (4). \square

In the situation envisaged in (2) and (4) we say that N_0 and N_1 are *in pairing* (cf. Definition 7.1).

Theorem 4.8 implies:

Corollary 4.10. If N_0 is an isolated center, then the leaves around it are $(m-1)$ -spheres, and if $\partial\mathcal{C}(N_0, \mathcal{F}) \neq \emptyset$ then the interior of $\mathcal{C}(N_0, \mathcal{F})$ is an m -ball D^m . Similarly (since every circle in an oriented manifold has trivial normal bundle), if N_0 has dimension 1 then every nearby leaf

is diffeomorphic to $S^1 \times S^{m-2}$ and if $\partial\mathcal{C}(N_0, \mathcal{F}) \neq \emptyset$ then the interior of $\mathcal{C}(N_0, \mathcal{F})$ is a product $S^1 \times D^{m-1}$.

Remark 4.11. When the foliation is also (transversely) real analytic, one can show that if there is a compact leaf with finite holonomy, then \mathcal{F} is closed. To see this, let $L \in \mathcal{F}$ be a compact leaf with finite holonomy, denote by Ω the union of all compact leaves of \mathcal{F} in M and let $\Omega(L) \subset \Omega$ be its connected component containing L . As we know already, if a leaf $L_1 \in \mathcal{F}$ is such that L_1 does not accumulate on a component of $\text{sing}(\mathcal{F})$ and it is in the boundary of $\Omega(L)$, then L_1 is compact. The holonomy group of L_1 is a subgroup of real analytic diffeomorphisms $\text{Hol}(L_1, \mathcal{F}) < \text{Diff}^w(\mathbb{R}, 0)$. Because L_1 is accumulated by compact leaves the holonomy group $\text{Hol}(L_1, \mathcal{F})$ has finite orbits arbitrarily close to the origin. This implies that $\text{Hol}(L_1, \mathcal{F})$ is finite and therefore trivial. Since L_1 is compact and has trivial holonomy it follows from the local stability theorem that we actually have $L_1 \subset \Omega(L)$, a contradiction. Thence every leaf in the boundary $\partial\Omega(L)$ must accumulate on $\text{sing}(\mathcal{F})$ and therefore it has to be a separatrix of some saddle singularity. Also this leaf is closed off $\text{sing}(\mathcal{F})$. Thus, every compact invariant subset $\Lambda(N) \subset \partial\Omega(L)$, obtained as the union of a saddle N and some of its separatrices, must have trivial holonomy. Thus all leaves nearby $\Lambda(N)$ are compact. This shows that $\partial\Omega$ is union of sets of the form $\Lambda(N)$ as above and since M is connected we conclude that $M = \Omega \cup \partial\Omega$, which implies that \mathcal{F} is a closed foliation.

5 Distinguished neighborhoods of saddles

We now look at the topology of the foliation near a saddle-type component of the singular set. We discuss first the isolated singularity case since this gives the local model. We begin with a fast review of classical Morse theory.

5.1 Isolated saddle

Let us assume that near a saddle point $p \in M$ the foliation is given by the fibers of a Morse function $f: U \rightarrow \mathbb{R}$ with Morse index r at x . That is, we can choose local coordinates where x is identified with the origin $0 \in \mathbb{R}^m$ and f is:

$$f(x_1, \dots, x_m) = \sum_{j=1}^r -x_j^2 + \sum_{j=r+1}^m x_j^2 \quad , \quad m > r > 0. \quad (1)$$

Notice that the local separatrix of \mathcal{F} union the singular point 0 is the hypersurface V given locally by $f^{-1}(0)$:

$$V = \left\{ \sum_{j=1}^r x_j^2 = \sum_{j=r+1}^m x_j^2 \right\}.$$

This is a cone, union of lines passing through the origin. Hence its topology is determined by its *link* $K = V \cap S_\epsilon$ where S_ϵ is a sphere around 0 (see [14]), which may actually be taken to be the unit sphere, that we denote \mathbb{S} and we think of it as bounding the unit ball \mathbb{D} . It is an exercise to show that K is diffeomorphic to the product $S^{r-1} \times S^{m-r-1}$ and therefore the separatrix $V \setminus \{0\}$ is $S^{r-1} \times S^{m-r-1} \times \mathbb{R}$ (in fact its intersection with U , assuming this set is small enough).

Notice that if r is 1 or $m-1$, and only in these cases, the link K has two connected components, otherwise it is connected. This is because in these two cases V actually has two “branches” (or components) that meet at 0. This means that if $r = 1, m-1$, then the saddle has two local separatrices. Thus in these cases, and only in these cases, an isolated saddle can have either one or two global separatrices, since both local separatrices can belong to the same global leaf.

The hypersurface V splits \mathbb{R}^m in two regions, corresponding to the points $x \in U$ where $f(x)$ is positive or negative (for $r = 1, m-1$ one actually has three regions, an “external one” and two “internal regions” bounded by the two components of V). Let us look at the fibers $V_t = f^{-1}(t)$ of f in these two cases.

We observe that V meets the unit sphere \mathbb{S} transversally. Thus, essentially by Thom’s transversality, for $t \neq 0$ sufficiently small one has that V_t also meets \mathbb{S} transversally. Then the first Thom-Mather Isotopy Theorem implies that for $t \neq 0$ sufficiently small, V and V_t are isotopic away from \mathbb{D} , so they only differ inside the ball; the V_t ’s also differ only inside the ball. In fact, to get V_t all one has to do, up to diffeomorphism, is to take the piece of V that is outside the interior of the unit ball, $V^* = V \setminus \overset{\circ}{\mathbb{D}}$, which has boundary $K = S^{r-1} \times S^{m-r-1}$, and attach to it either $D^r \times S^{m-r-1}$ to get the fibers V_t for $t > 0$, or $S^{r-1} \times D^{m-r}$ to get the fibers V_t for $t < 0$.

In other words, if t^+ is positive and t^- is negative, then the fibers V_{t^+} and V_{t^-} are obtained from each other by the classical Milnor-Wallace surgery (see [12]), *i.e.*, by removing $D^r \times S^{m-r-1}$ from V_{t^+} to get a manifold with boundary $S^{r-1} \times S^{m-r-1}$, and then attaching $S^{r-1} \times D^{m-r}$ to get V_{t^-} , and viceversa.

In the sequel we need to consider *distinguished neighborhoods* of saddle singularities. This means a set $W(N_0)$, homeomorphic to an m -ball, of the form:

$$W(N_0) = \{(x_1, \dots, x_m) \in B_\epsilon \mid -\delta \leq f(x_1, \dots, x_m) \leq \delta\},$$

where $B_\epsilon \subset M$ is diffeomorphic to a ball. Thus $W(N_0)$ is bounded by the leaves $f^{-1}(\pm\delta)$ and the sphere $S_\epsilon = \partial B_\epsilon$, with δ small enough with respect to ϵ , so that all the fibers $f^{-1}(t)$ with $|t| \leq \delta$ meet ∂B_ϵ transversally.

In other words, $W(N_0)$ can be regarded as a *Milnor tube* for f at N_0 (see the last chapter of [14]). Its boundary $\partial W(N_0)$ is homeomorphic to an $(m-1)$ -sphere and consists of three “pieces”: two leaves of \mathcal{F} and the *cap* denoted \mathcal{C} consisting of the points $x \in S_\delta$ with $|f(x)| < \delta$:

$$\partial W(N_0) = [(f^{-1}(-\epsilon) \cup f^{-1}(\epsilon)) \cap B_\delta] \cup \mathcal{C}. \quad (2)$$

At each point in \mathcal{C} the corresponding leaf of \mathcal{F} is transversal to $\partial W(N_0)$. Notice that we speak of three “pieces” in $\partial W(N_0)$, each of which may have one or two connected components, depending on the Morse index of the corresponding saddle.

5.2 Non-isolated saddle

Let N_0 be now a saddle singularity of dimension $n_0 > 0$. Given a point $p \in N_0$, define a *flow box* for \mathcal{F} at p to be a set of the form $W = \Sigma \times D^{n_0}$ where:

- i) Σ is an $(m-n_0)$ -disc transversal to N_0 at p , with a Morse foliation defined by the intersection of Σ with the leaves of \mathcal{F} , singular at p ; and
- ii) restricted to W the foliation \mathcal{F} is a product foliation.

We say that W is a *distinguished flow box* if the transversal Σ is a distinguished neighborhood of the isolated saddle p .

Definition 5.1. A *distinguished neighborhood* of N_0 is a compact neighborhood $W(N_0)$ of N_0 , which is union of a finite collection W_1, \dots, W_s of distinguished flow boxes for points in N_0 , such that:

1. The intersection of any two of them is either empty or a flow box.
2. $W(N_0)$ can be identified with the normal bundle of N_0 and therefore $\mathcal{F}|_{W(N_0)}$ has locally a product structure with an isolated saddle singularity in each normal fibre. Hence each normal fibre inherits a decomposition in three pieces as in equation (2).
3. The boundary of $W(N_0)$ is union of leaves of \mathcal{F} and a fibre bundle $\tilde{\mathcal{C}}$ over N_0 with fiber the cap \mathcal{C} in equation (2), consisting of points where the foliation is transversal to $\partial W(N_0)$.

The existence of distinguished neighbourhoods for closed Bott-Morse foliations is granted by the local product structure at each component of the singular set, the triviality of the holonomy, and the compactness of the singular set.

Proposition 5.2. *Let N_0 be saddle-type component of the singular set of a transversally oriented, codimension 1 closed foliation \mathcal{F} on the closed, oriented, smooth manifold M . Let $W(N_0)$ be a distinguished neighborhood of N_0 . Then for each leaf L of \mathcal{F} that meets $W(N_0)$, one has that the intersection $L \cap W(N_0)$ is a fiber bundle over N_0 with fiber $L \cap \Sigma$, the trace of L in a transversal Σ .*

The proof is obvious.

Remark 5.3. It is not truth in general that the leaves $L \cap W(N_0)$ have a global product structure, even if the normal bundle of N_0 is trivial. For instance, take a 2-disc D in \mathbb{R}^2 with a saddle singularity at 0; now take in \mathbb{R}^3 the product $D \times [0, 2\pi]$. As you move upwards from the level $D \times \{0\}$ a time t , rotate the disc by an angle t . Hence at the level $D \times \{1\}$ the foliation on the disc is exactly as in the level $D \times \{0\}$, so we can glue these two 2-discs and get a foliation on a solid torus $D \times S^1$ with trivial holonomy; no leaf has a global product structure. Nevertheless, this is not surprising since already in the nonsingular case a compact leaf with trivial holonomy or homotopy does not give a global product structure for the foliation, but a fibre bundle structure.

6 The Partial stability theorem

In this section we give one of the main results in this work, the Stability Theorem 6.3. This is analogous to the classical Partial Stability Theorem of Reeb for compact leaves, and to the Partial Stability Theorem

4.3 for center-type components of Bott-Morse foliations. Then we look at the global topology of separatrices and use the Stability Theorem to describe the topology of the nearby leaves.

6.1 The Partial stability theorem

As in the classical case of non-singular foliations with trivial holonomy, using also the product structure of \mathcal{F} around N_0 we obtain:

Proposition 6.1. *Let \mathcal{F} be a closed Bott-Morse foliation on a compact manifold M , $N_0 \subset \text{sing}(\mathcal{F})$ a saddle component and $\Lambda(N_0)$ be the union of N_0 and all its separatrices. Consider a fundamental system of compact invariant neighborhoods $\{W_\nu\}$ of $\Lambda(N_0)$ in M as in Theorem 3.2, and a distinguished neighborhood $U(N_0)$ of N_0 . Then for each $\nu \gg 1$ one has that $W_\nu \setminus (W_\nu \cap U(N_0))$ fibers over $\Lambda(N_0) \setminus (\Lambda(N_0) \cap U(N_0))$ by transverse segments Σ_x , $x \in \Lambda(N_0) \setminus U(N_0)$, so that for every leaf L close enough to $\Lambda(N_0)$ the intersection $L \cap \Sigma_x$ is a single point.*

Proof. This is a consequence of the proof of Reeb's local stability theorem in [2] (see Lemma 6 and Theorem 4 in Chapter 4). By Theorem 3.2, there is a fundamental system of distinguished neighborhoods $\{V_\alpha\}$ of N_0 such that $\mathcal{F}|_{V_\alpha}$ is given by a Bott-Morse function f_α with singular set N_0 .

Denote by Λ_j , $j = 1, \dots, r$ the connected components of $\Lambda(N_0) \setminus N_0$. Given a distinguished neighborhood $V = V_\alpha$ of N_0 we set $\tilde{\Lambda}_j = \Lambda_j \setminus (\Lambda_j \cap V)$. The $\tilde{\Lambda}_j$ are the connected components of $\Lambda(N_0) \setminus (\Lambda(N_0) \cap V)$. Set $\tilde{\mathcal{F}} := \mathcal{F}|_{M \setminus V}$. The leaves of $\tilde{\mathcal{F}}$ are closed with trivial holonomy. Thus by the same arguments as in the proof of Reeb's stability theorem, there is a fundamental system of invariant neighborhoods $\{\tilde{W}_\nu^{(j)}\}_\nu$ of $\tilde{\Lambda}_j$ where $\tilde{\mathcal{F}}$ is equivalent to a product foliation on $\tilde{\Lambda}_j \times (-1, 1)$ with leaves of the form $\tilde{\Lambda}_j \times \{t\}$, $t \in (-1, 1)$.

Moreover, there is a fibration of $\tilde{W}_\nu^{(j)}$ over $\tilde{\Lambda}_j$ by transverse segments meeting each leaf that intersects $\tilde{W}_\nu^{(j)}$ at exactly one point. Considering the unions $W_\nu := (\bigcup_j \tilde{W}_\nu^{(j)}) \cup V$ we obtain compact invariant neighborhoods of $\Lambda(N_0)$ as in the statement. \square

Definition 6.2. Let $N_0 \subset \text{Sad}(\mathcal{F})$ be a saddle component and denote by $\Lambda(N_0)$ the union of N_0 and its separatrices. We say that \mathcal{F} has *order one over $\Lambda(N_0)$* if there is a distinguished neighborhood $U(N_0)$ of N_0

and a fundamental system of compact invariant neighborhoods W_ν of $\Lambda(N_0)$, such that:

- (1) For each W_ν one has that $W_\nu \setminus (W_\nu \cap U(N_0))$ fibers over $\Lambda(N_0) \setminus (\Lambda(N_0) \cap U(N_0))$ by transverse segments Σ_x , $x \in \Lambda(N_0) \setminus U(N_0)$; and
- (2) for every leaf L close enough to $\Lambda(N_0)$ the intersection $L \cap \Sigma_x$ is a single point.

In this case each neighborhood W_ν is called a *bundle neighborhood* of $\Lambda(N_0)$ with respect to \mathcal{F} .

The following result is essential in our work.

Theorem 6.3 (Partial Stability Theorem). *Let \mathcal{F} be a closed Bott-Morse foliation on a manifold M and $N \subset \text{sing}(\mathcal{F})$ a center-type component with basin $\mathcal{C}(N, \mathcal{F})$. Let $N_0 \subset \partial\mathcal{C}(N, \mathcal{F})$ be a saddle type component and let $V \subset M$ be a distinguished neighborhood of N_0 . Let $\Lambda(N_0) \subset \partial\mathcal{C}(N, \mathcal{F})$ be the union of N_0 and a separatrix and Λ_0 a connected component of $\Lambda(N_0) \setminus (\Lambda(N_0) \cap V)$. Then there is a fundamental system of neighborhoods $\{W_\alpha\}$ of Λ_0 , each contained in a bundle neighborhood W_ν , such that if L_i, L_e are interior and exterior leaves of \mathcal{F} intersecting W_α with $L_i \subset \mathcal{C}(N, \mathcal{F})$ and $L_e \cap \mathcal{C}(N, \mathcal{F}) = \emptyset$, then $L_i \cap W_\alpha$ and $L_e \cap W_\alpha$ are homeomorphic to Λ_0 .*

Proof. The proof is a straightforward consequence of Proposition 6.1. Let $\Lambda(N_0)$ be the union of N_0 with all its separatrices. Consider the maps $\phi_i: \Lambda_0 \rightarrow L_i$, $x \mapsto \phi_i(x) := L_i \cap \Sigma_x$ and $\phi_e: \Lambda_0 \rightarrow L_e$, $x \mapsto \phi_e(x) := L_e \cap \Sigma_x$. Then we define $\phi: L_i \rightarrow L_e$ as $\phi = \phi_e \circ \phi_i^{-1}$. \square

6.2 Topology of leaves near compact invariant sets with saddles

As before, let N be a saddle-type component of the singular set of a foliation \mathcal{F} with Bott-Morse singularities, and let Λ be the union of N and all its separatrices. We want to use the results of the previous sections to compare the topology of the leaves near Λ with that of Λ itself, so we say first a few words about the latter. We start with the case of an isolated saddle.

Assume $N = \{0\}$ is an isolated saddle with Morse index $r = 1$ or $r = m - 1$. Since both cases are similar we discuss only the case $r = 1$. As mentioned earlier, in this case there are two local separatrices,

corresponding to the two branches of the hypersurface

$$V = \{x_1^2 = \sum_{j=2}^m x_j^2\}.$$

The saddle is the point 0 and the set $V \setminus \{0\}$ consists of two connected components. It can happen that both components belong to the same leaf or that they belong to different leaves. In the first case Λ consists of a single leaf L , compactified by attaching to it the saddle singularity 0; this situation is depicted in figure 11 below. One has a self-saddle-connection.

If the two components of $V \setminus \{0\}$ belong to different leaves, then Λ consists of these two leaves union the saddle point $\{0\}$, and there can not be more separatrices since every separatrix must contain a local separatrix, and there are only two of them.

When the Morse index of the saddle at 0 is not 1 nor $m - 1$, then the local separatrix is connected, so there can only be one global separatrix L , so $\Lambda = L \cup \{0\}$.

Similar statements obviously hold for a non-isolated saddle N of dimension n : if we look at the restriction to a distinguished neighborhood of N we see that the separatrix (or separatrices) is a fibre bundle over N with fibre the trace in a transversal. Hence, if its (transverse) Morse index is 1 or $m - n - 1$, then one has two local separatrices, which may belong to the same leaf of \mathcal{F} or to two different leaves, and this is independent of the choice of point in N . If the Morse index is not 1 nor $m - n - 1$, then one has only one local separatrix, which must belong to a single leaf.

Summarizing one has:

Lemma 6.4. *Let N be an n -dimensional, $n \geq 0$, saddle-type component of the singular set of \mathcal{F} , and let Λ be the union of N and all its separatrices. Then:*

i) *If the Morse index of N is not 1 nor $(m - n - 1)$, then N has only one local separatrix. Hence Λ consists of N and one single leaf L of \mathcal{F} : $\Lambda = L \cup \{N\}$.*

ii) *If the Morse index of N is 1 or $(m - n - 1)$, then N has two local separatrices, which may or may not belong to the same leaf. Hence Λ may consist of N and one single leaf L , or N union two leaves.*

Notice that since \mathcal{F} is assumed to be proper, of codimension 1 and transversally oriented, if we denote by $\overset{\circ}{W}(N)$ the interior of a distin-

guished neighborhood of N , then $\Lambda \setminus \overset{\circ}{W}(N)$ is a compact, codimension 1, oriented submanifold of M , consisting of either one or two components, leaves of \mathcal{F} restricted to $(M \setminus \overset{\circ}{W}(N))$.

Let us now look at the leaves of \mathcal{F} near Λ . Since the foliation is closed and has Bott-Morse singularities, we can choose a distinguished neighborhood $W(N)$ small enough so that every leaf of \mathcal{F} that meets $W(N)$ of N is compact. This condition is satisfied whenever N is in the boundary of the basin of some center, by theorem 6.3.

The case of an isolated saddle N is now rather simple: assume first Λ consists of N and one single leaf L_0 . L_0 is oriented, of codimension 1, it splits M in two connected components that we denote by M^+ and M^- . By theorem 6.3, away from a distinguished neighborhood $W(N)$, the leaves of \mathcal{F} in either side M^+ and M^- are homeomorphic to $\Lambda \setminus W(N)$. Then, from the discussion for isolated saddles in Section 5 we see that if L^+ is a leaf in M^+ that meets $W(N)$ and N has Morse index r , then $L^+ \setminus W(N)$ has boundary $S^{r-1} \times Sm - r - 1$, which is also the boundary of $\Lambda \setminus W(N)$ and the boundary of $L^- \setminus W(N)$ for leaves in M^- .

To recover L^+ up to homeomorphism, we attach $D^r \times Sm - r - 1$ to $\Lambda \setminus W(N)$ along their common boundary; to get L^- we attach $S^{r-1} \times Dm - r$ to $\Lambda \setminus W(N)$ along their common boundary; and to get Λ back we simply collapse the boundary of $L^- \setminus W(N)$ to a point.

In other words, the leaves L^+ and L^- are obtained from each other by Milnor-Wallace surgery ([12]).

If the Morse index r of N is 1 or $m - 1$, what we are doing is that we remove from one leaf L^+ a copy of $S^{m-2} \times D^1$, where $D^1 = I$ is an interval, so one has boundary $S^{m-2} \times S^0$, *i.e.*, two $(m - 2)$ -spheres, and then glue back a copy of $D^{m-1} \times S^0$, *i.e.*, two $(m - 1)$ -discs, or viceversa. In other words, L^+ is obtained from L^- by attaching to it a 1-handle.

If the Morse index r of N is 1 or $m - 1$ and Λ consists of N and two leaves L_1 and L_2 , then all the previous discussion applies in exactly the same way, the only difference is that now $L^- \setminus W(N)$ has two connected components, homeomorphic to $(L_1 \cup L_2) \setminus W(N)$, and L^+ is the connected sum of these two manifolds.

Now consider a saddle component N of dimension $n > 0$ and (transverse) Morse index r , $m - n > r > 0$. Let $W(N)$ be an open distinguished neighborhood of N , sufficiently small so that every leaf that meets $W(N)$ is compact. Such a neighborhood exists because the foliation is proper, there are no saddle-connections, there are only finitely many components of the the singular set of \mathcal{F} , and at each component

there are at most two separatrices.

As in the isolated singularity case, Λ splits M in two connected components that we denote by M^+ and M^- . By theorem 6.3, away from $W(N)$, the leaves of \mathcal{F} in either side M^+ and M^- are homeomorphic to $\Lambda \setminus W(N)$. From the discussion in Section 5 we see that if L^+ is a leaf in M^+ that meets $W(N)$, then $L^+ \setminus W(N)$ has boundary K which is diffeomorphic to an $(S^{r-1} \times S^{m-n-r-1})$ -bundle over N ; up to homeomorphism, this is also the boundary of $\Lambda \setminus W(N)$, and the boundary of $L^- \setminus W(N)$ for leaves in M^- sufficiently near Λ .

To recover L^+ up to homeomorphism, we attach to $\Lambda \setminus W(N)$ the corresponding bundle with fibre $(D^r \times S^{m-n-r-1})$, *i.e.*, we “fill in” the $(r-1)$ -sphere in each fiber; to get L^- we “fill in” the $(m-n-r-1)$ -sphere in each fiber, *i.e.*, we attach the corresponding $(S^{r-1} \times D^{m-n-r})$ -bundle to $\Lambda \setminus W(N)$ along their common boundary; and to get Λ back we collapse to a point each $(S^{r-1} \times S^{m-n-r-1})$ -fiber in the boundary of $L^- \setminus W(N)$.

We summarize the previous discussion in the following theorem:

Theorem 6.5. *Let \mathcal{F} be a closed Bott-Morse foliation on M^m . Let N be a saddle-type component of dimension $n \geq 0$ and (transverse) Morse index r , $m-n > r > 0$. Let $W(N)$ be a sufficiently small distinguished open neighborhood of N such that every leaf of \mathcal{F} that meets $W(N)$ is compact. Let Λ be the union of N and its separatrices. Then:*

i) The set Λ consists of N and at most two leaves of \mathcal{F} . Moreover, if $r \neq 1, m-n-1$, then there is exactly one leaf of \mathcal{F} in Λ .

ii) If L is a leaf of Λ sufficiently near Λ at some point, then $L \setminus W(N)$ is a compact manifold with boundary K homeomorphic to the boundary of $\Lambda \setminus W(N)$.

iii) The manifold K is a fibre bundle over N with fiber $(S^{r-1} \times S^{m-n-r-1})$.

iv) The leaves of \mathcal{F} near Λ which are contained in different connected components of $M \setminus \Lambda$ are obtained from each other by performing Milnor-Wallace surgery fiberwise. More precisely, up to homeomorphism, to get the leaves in one side we attach to $\Lambda \setminus W(N)$ the obvious bundle with fiber $(D^r \times S^{m-n-r-1})$, and to get the leaves in the other side we attach the corresponding bundle with fiber $(S^{r-1} \times D^{m-n-r})$. To recover Λ we just collapse to a point each $(S^{r-1} \times S^{m-n-r-1})$ -fiber in K .

In the sections below we give examples of isolated saddles on 3-manifolds with Morse index 1 having only one global separatrix and also examples having two separatrices.

7 Pairings and Foliated surgery

Let us motivate what follows by recalling the classical center-saddle bifurcation in the plane, which is depicted in Figure 5. We have an isolated center in the plane and a saddle in the boundary of its basin. These are replaced via an isotopy by a trivial foliation. This elimination procedure may be described as follows. Denote by $Z_\varepsilon = (x_1^2 - \varepsilon)\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$, $\varepsilon > 0$. This vector field has a pair of singularities saddle-source for $\varepsilon > 0$, a saddle-node singularity for $\varepsilon = 0$ and no singularity for $\varepsilon < 0$. Thus the original pairing center-saddle can be viewed as a deformation of a trivial vertical foliation via passing through a saddle-node singularity.

Other center-saddle arrangements in dimension two are depicted in Figure 8. The figure on the left shows a disc with a center-singularity which is replaced (or replaces, depending on the orientation of the arrow) a disc with three singularities: two of them are centers and one is a saddle which is in the boundary of the basin of both centers. The figure on the right also shows a saddle in the boundary of the basins of two centers. Notice that there is a significant difference in these two cases. In the first case the saddle has only one separatrix, while in the second case it has two separatrices.

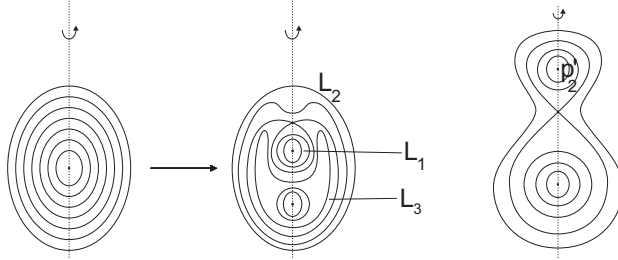


Figure 8:

More generally,

Definition 7.1 (Pairing). We say that a saddle component $N_0 \subset \text{sing}(\mathcal{F})$ and a center component $N_1 \subset \text{sing}(\mathcal{F})$ are *in pairing*, denoted $N_0 \leftrightarrow N_1$, if N_0 is in the boundary of the basin of N_1 , *i.e.*, $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F})$. If N_0 is in pairing with several center components N_1, \dots, N_r we denote it by $N_0 \leftrightarrow (N_1, \dots, N_r)$.

Definition 7.2 (Foliated Surgery). Let \mathcal{F} be a Bott-Morse foliation on M^m , and let N_0 be a saddle component which is in pairing with some center components N_1, \dots, N_r , $r \geq 1$. A *foliated surgery* for the pairing $N_0 \leftrightarrow (N_1, \dots, N_r)$ means a choice of a compact, connected region $R \subset M$ containing N_0, N_1, \dots, N_r in its interior $\overset{\circ}{R}$, such that:

1. The restriction of the foliation \mathcal{F} to R can be replaced by a Bott-Morse foliation \mathcal{F}_m on R which coincides with \mathcal{F} in a neighborhood of the boundary of R ;
2. The new foliation on M , also denoted \mathcal{F}_m , is Bott-Morse;
3. If \mathcal{F} is proper, so is \mathcal{F}_m .

Foliated surgery can be used to reduce the singularities of Bott-Morse foliations. The following definitions make this idea precise.

Definition 7.3 (Reducible and trivial pairings). Let \mathcal{F} be as before, a closed Bott-Morse foliation on a connected closed oriented m -manifold M .

1. We say that a pairing $N_0 \leftrightarrow (N_1, \dots, N_s)$ is *trivial* if there is a foliated surgery for this pairing in a region R , such that the resulting foliation is regular on R . In this case we say that the new foliation \mathcal{F}_m is obtained from \mathcal{F} by *eliminating* the components N_0, N_1, \dots, N_s .
2. We say that a pairing $N_0 \leftrightarrow (N_1, \dots, N_s)$ is *reducible* if there is a foliated surgery for it in a region R , such that the singular set of the resulting foliation \mathcal{F}_m has at most s components in R . In this case we say that \mathcal{F}_m is obtained from \mathcal{F} by *replacement* or *reduction* of the components N_0, N_1, \dots, N_s .
3. A pairing which is not reducible is called *irreducible*.

Remark 7.4. We will see later that the type of surgeries we perform for proving Theorem 9.1 also satisfy the following condition:

4. The holonomy pseudogroup of \mathcal{F}_m is obtained from the holonomy pseudogroup of \mathcal{F} by *reduction* in the following sense:

Definition 7.5. We say that the holonomy pseudogroup of \mathcal{F}_m is *obtained by reduction* from the holonomy pseudogroup of \mathcal{F} if there is a collection $\mathcal{T} = \{T_j\}_{j \in J}$ of sections $T_j \subset M$ transverse to \mathcal{F}_m such that:

- i) Each leaf of \mathcal{F}_m intersects some T_j ;
- ii) Each T_j is also transverse to \mathcal{F} , though it may happen that not all leaves of \mathcal{F} intersect the union $\bigcup_{j \in J} T_j$;
- iii) The holonomy pseudogroup $\text{Hol}(\mathcal{F}_m, \mathcal{T})$ of \mathcal{F}_m with respect to \mathcal{T} is isomorphic to the subpseudogroup of the holonomy pseudogroup of \mathcal{F} generated by the holonomy maps corresponding to paths γ contained each in some leaf L of \mathcal{F} and joining a point in $T_i \cap L$ to a point in $T_j \cap L$.

7.1 Pairings in dimension three

We now introduce certain types of possible center-saddle pairings that appear in dimension three for closed Bott-Morse foliations \mathcal{F} . These give all possible pairings satisfying the conditions of Theorem 9.1.

7.1.1 Bundle-type pairings

We have two different bundle-type pairings obtained by "lifting" two-dimensional pictures to a product $S^1 \times D^2$, where D^2 is homeomorphic to a 2-disc. We recall that every oriented 2-disc bundle over S^1 is trivial and, moreover, every circle embedded in an oriented manifold has trivial normal bundle.

Just as in 5.3, the foliations we give on $S^1 \times D^2$ are locally products of a 2-dimensional picture by an interval, but they may not be products globally, though for simplicity in the pictures below we always represent the product foliations. Pairing (P.NI.1) (abbreviation for "product-nonisolated" of type 1) is depicted in Figure 9 and consists of an S^1 -saddle component and an S^1 -center component. This pairing is trivial.

The second bundle-type pairing is (P.NI.2) (abbreviation "product-nonisolated" of type 2), obtained as the lift to $S^1 \times D^2$ of a pairing as in Figure 10 below. Here an S^1 -saddle component is in pairing with two S^1 -center components. One of these is the previous trivial pairing, the other is the non-trivial pairing (P.NI.2).

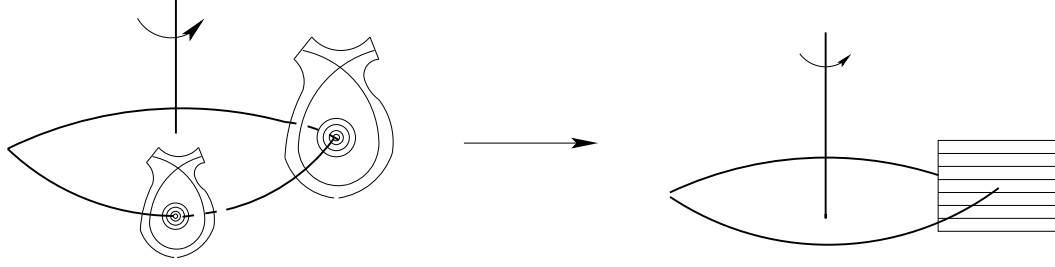


Figure 9: P.NI.1

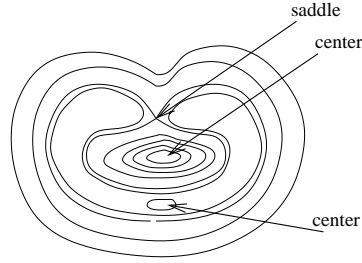


Figure 10: P.NI.2

7.1.2 Isolated singularities

The list of possible pairings of isolated singularities in dimension 3 is given in [3]. The first is a trivial pairing (T.I) (for “trivial-isolated”) as in Example 2.11. This is obtained as rotation of the two-dimensional picture in Figure 5 with respect to the vertical axis. The second, (NT.I.1) (for nontrivial-isolated of type 1) is depicted on the left in Figure 11; it is a non-trivial pairing with spherical leaves inside the region bounded by the separatrices and tori outside. On the right in Figure 11 we have a non-trivial, isolated pairing of type 2 (NT.I.2), obtained by rotation of the image with respect to the *horizontal* axis. The pairing consists of an isolated saddle at the origin and a center *at infinity*. The outer and the inner leaves are 2-spheres and the picture can be completed by two isolated centers (on the left and on the right of the picture) each one of them defining a trivial pairing of type (T.I) with the saddle.

Another rotation type pairing is depicted in Figure 12. Here we start with an isolated center C_0 at the origin in a two dimensional

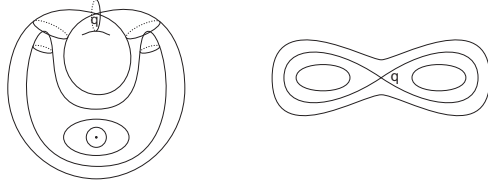


Figure 11: Pairings NT.I.1 and NT.I.2

picture. Then we introduce a (two-dimensional) center saddle pairing of type trivial isolated, and we rotate it with respect to the vertical axis. The pairing we consider is the one given by the original center C_0 and the saddle. This pairing is non trivial and will be called (NT.I.3) (for "nontrivial-isolated" of type 3).

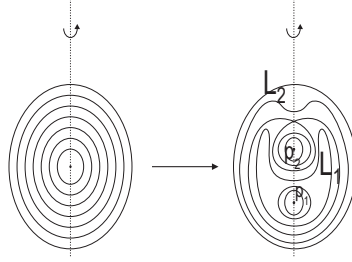


Figure 12: Pairing (NT.I.3)

Notice that the 2-dimensional picture of this pairing is like in Figure 10. The difference is that for that pairing we take the product with S^1 , while here we are making the 2-dimensional model rotate with respect to an axes as indicated in Figure 12.

7.1.3 Non-isolated saddle

Rotation of Figure 13 with respect to the vertical axis gives two isolated centers in pairing with a non-isolated saddle. These will be called (U.NI.) (for "upper-nonisolated") and the other center-saddle pairing will be called (L.NI.) (for "lower-nonisolated").

One also has a product type pairing (P.NI.) (for "product-nonisolated") obtained by rotating the Figure 14 with respect to the vertical axis. The pairing we consider consists of the non-isolated center and the non-

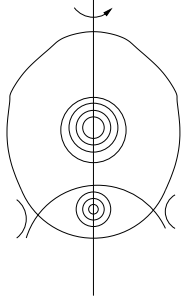


Figure 13: Pairings $(U.NI.)$ and $(L.NI.)$.

isolated saddle.

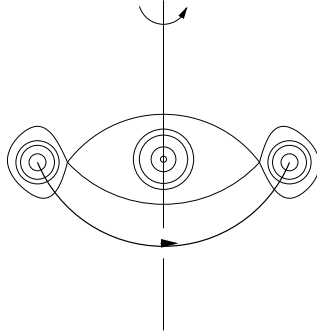


Figure 14: Pairing $(P.NI.)$

7.2 Irreducible components

We describe the irreducible pairings in dimension three that appear in the situation we study, *i.e.*, \mathcal{F} closed Bott-Morse foliation with $c(\mathcal{F}) > 2s(\mathcal{F})$:

1 - Disconnected irreducible component of bitorus type. We consider an isolated saddle N_0 and two non-isolated centers N_1, N_2 as singular set of a foliation in a solid torus as described below. The union $\Lambda(N_0)$ of the saddle and its separatrices is homeomorphic to a bitorus obtained by gluing three pieces P_1, P_2, C where each P_j is a torus minus a disc, C is a cylinder, and then collapsing one of the boundary

curves of C into a point N_0 . The outside leaves are diffeomorphic to the bitorus (see figure 15). The complement $\Lambda(N_0) \setminus \{N_0\}$ has two connected components Λ_1, Λ_2 such that $\Lambda_j \cup \{N_0\}$ is homeomorphic to a torus pinched at the point and bounds a region R_j foliated by tori. We have a center $N_j \subset R_j$ such that $\mathcal{C}(N_j, \mathcal{F}) = R_j$ and $\partial\mathcal{C}(N_j, \mathcal{F}) = \Lambda_j$. We call this case *disconnected irreducible* case, because $\Lambda(N_0) \setminus N_0 = \Lambda_1 \uplus \Lambda_2$ is the disjoint union of two components. Notice this is a special case of Example 2.9.

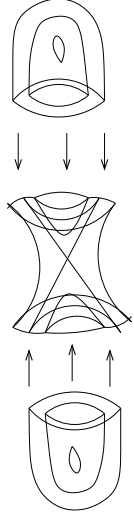


Figure 15: Irreducible component of bitorus type.

2 - Torus-ball type component. A second model is obtained with a torus T bounding a solid torus Ω in \mathbb{R}^3 . Inside Ω we foliate a neighborhood of $\partial\Omega = T$ by concentric tori and introduce an isolated saddle singularity N_0 by collapsing the boundary of a disc into a point N_0 so that the resulting variety $\Lambda(N_0)$ is the union of a 2-sphere and a torus, both pinched at N_0 (see figure 16). Now we foliate the region interior to the sphere by spheres and the region interior to the torus by tori accumulating to a circle N_1 . The exterior of Ω can be foliated by tori centered at a circle at the infinity N_2 . Notice that one of the pairings is trivial of type (T.I). This gives our second model of "disconnected singularity".

Definition 7.6. We call the two pairings constructed above *discon-*

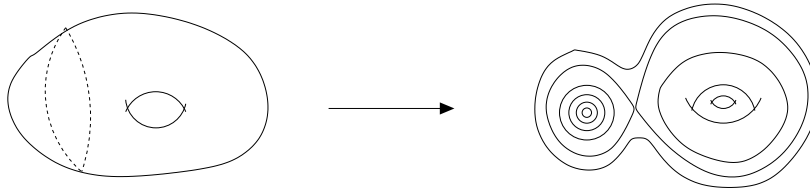


Figure 16: Disconnected component of torus ball type.

nected pairings. (The name indicates the fact that the closure of the union of separatrices is disconnected if we remove the saddle.)

3 - Connected irreducible component (only one center). We consider an isolated saddle N_0 and a non-isolated center N_1 as singular set of a foliation in a solid bitorus as depicted below (see figure 17). We begin with a solid torus region $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ a bitorus. Foliate a neighborhood of the boundary by bitorus and fix one of these bitorus; take a meridian in one of the handles and collapse it to a point N_0 . Call this singular leaf $\Lambda(N_0)$. Foliate now the interior of the solid region bounded by Λ by tori in such a way that we have a single component of the singular set, which is a one-dimensional center N_1 . The leaves outside Λ are diffeomorphic to the bitorus and the inner leaves are tori which accumulate to N_1 . We call this a *connected irreducible* case. Indeed, $\Lambda(N_0)$ is the union of N_0 with all the separatrices accumulating to N_0 and $\Lambda(N_0) \setminus N_0$ is connected.

4 - Multiple irreducible connected component. Let $L(g)$ be an unknotted solid handle-body in \mathbb{R}^3 with genus $g > 1$, and foliate $L(g)$ as in Example 2.9. We first have leaves of genus g , parallel to the boundary. Then a surface S with $g - 1$ saddle-points which bounds in g -components, each diffeomorphic to an open solid torus T_j , $j = 1, \dots, g$. Then we foliate each T_j in the usual way, by copies of $S^1 \times S^1$, having in each torus a circle N_j as singular set, all of center-type.

8 Proper Bott-Morse foliations on 3-manifolds

From now on we assume the manifold M is 3-dimensional, and N_1, N_2 are distinct center-type components of $\text{sing}(\mathcal{F})$ such that $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) \neq \emptyset$. We thus know from Theorem 4.8 that there is exactly one saddle component N_0 in $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$, and Theorem 6.5

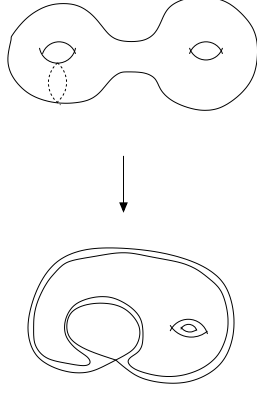


Figure 17: Connected irreducible component with only one center and original genus $g = 2$.

says that each $\partial\mathcal{C}(N_j, \mathcal{F})$ is the union of N_0 and separatrices of N_0 . This implies:

Lemma 8.1. *We can have one of the following possibilities:*

i) *The saddle N_0 has only one separatrix. In this case one has $\partial\mathcal{C}(N_1, \mathcal{F}) = \partial\mathcal{C}(N_2, \mathcal{F})$.*

ii) *The saddle N_0 has two separatrices: then $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ is a union of N_0 and separatrices of N_0 , and one can have:*

- $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) = N_0$.
- $\partial\mathcal{C}(N_1, \mathcal{F}) \subsetneq \partial\mathcal{C}(N_2, \mathcal{F})$ or viceversa, $\partial\mathcal{C}(N_2, \mathcal{F}) \subsetneq \partial\mathcal{C}(N_1, \mathcal{F})$, and $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ is N_0 union a separatrix.
- $\partial\mathcal{C}(N_1, \mathcal{F}) = \partial\mathcal{C}(N_2, \mathcal{F})$.

We now describe the possible arrangements for N_0, N_1, N_2 and the basins $\mathcal{C}(N_1, \mathcal{F}), \mathcal{C}(N_2, \mathcal{F})$. The possible cases are divided according to the following hierarchy:

- (1) the dimension of the saddle N_0 ,
- (2) the dimensions of the centers N_1, N_2 and
- (3) whether $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ is contained in $\text{sing}(\mathcal{F})$ or not, *i.e.*, whether $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) = N_0$ or not.

8.1 Isolated saddle

We begin with the case where $N_0 = \{q\}$, $N_1 = \{p_1\}$ and $N_2 = \{p_2\}$ are isolated singularities, studied in [3]. The following two results are respectively Lemma 3 and Lemma 2 in [3]:

Lemma 8.2 (isolated centers). *Let \mathcal{F} be a Morse foliation on a connected 3-manifold M . If an isolated saddle singularity q is such that $q \in \partial\mathcal{C}(p_j, \mathcal{F})$ for two isolated centers p_1, p_2 , then one of the pairings $q \leftrightarrow p_j$ is trivial.*

Lemma 8.3. *Let \mathcal{F} be a Morse foliation on a connected compact 3-manifold M , let $p \in \text{sing}(\mathcal{F})$ be an isolated center and q an isolated saddle contained in $\partial\mathcal{C}(p, \mathcal{F})$. Then we have the following possibilities:*

- (i) *If $\partial\mathcal{C}(p, \mathcal{F}) \setminus \{q\}$ is connected, then:*
 - (a) *either $\partial\mathcal{C}(p, \mathcal{F})$ is homeomorphic to a sphere S^2 with a pinch at q (a tear drop) and the pairing $q \leftrightarrow p$ is trivial; or*
 - (b) *$\partial\mathcal{C}(p, \mathcal{F})$ is homeomorphic to a pinched torus, obtained from a torus $S^1 \times S^1$ by collapsing a meridian to a point.*
- (ii) *If $\partial\mathcal{C}(p, \mathcal{F}) \setminus \{q\}$ has two connected components, then $\partial\mathcal{C}(p, \mathcal{F})$ is the union of two spheres S^2 which meet at q .*

An example of type (i.a) is the basin of the center p_2 in Figure 12, while the basin of p_1 in that same picture illustrates the case (i.b). An example of type (ii) is given in figure 12, corresponding to the pairing NT.I.1.

Next we have the case of an isolated saddle in pairing with two non-isolated centers.

Lemma 8.4 ($N_0 = \{q\}$, two non-isolated centers.). *Let N_1, N_2 be one-dimensional center-type components and N_0 an isolated saddle such that $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$. Assume further that \mathcal{F} satisfies the inequality $c(\mathcal{F}) > 2s(\mathcal{F})$. Then N_0, N_1, N_2 are in a same disconnected component of \mathcal{F} (as in Definition 7.6). This is of bitorus type if $N_0 = \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$, or of torus-ball type if otherwise.*

Proof. Let $U = U(N_0)$ be a distinguished neighborhood of N_0 . By Corollary 4.10 each leaf $L \subset \mathcal{C}(N_j, \mathcal{F})$, $j = 1, 2$, is a torus and the interior of the basin $\mathcal{C}(N_j, \mathcal{F})$ is a solid torus. The local separatrices of

the saddle N_0 divide the neighborhood U into three open regions $\overset{\circ}{R}_1, \overset{\circ}{R}_2, \overset{\circ}{R}_3$ as in Figure 7, glued along the separatrices. For each $j = 1, 2, 3$, denote by R_j the topological closure of $\overset{\circ}{R}_j$ in U .

Let us now assume that $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) = N_0$. Then, up to reordering the regions R_1 and R_2 , we have $\partial\mathcal{C}(N_j, \mathcal{F}) \cap U \subset R_j$ for $j = 1, 2$. In this case given a leaf $L_j \in \mathcal{C}(N_j, \mathcal{F})$ for $j = 1, 2$ we have that $L_j \setminus (L_j \cap U)$ is a torus minus a disc. This shows, by Theorem 6.3, that a leaf L of \mathcal{F} with $L \cap U \subset R_3$ is the union of two tori minus a disc in each, and a cylinder which corresponds to the hyperboloid leaf on R_3 ; hence L is a bitorus. This shows the existence of an invariant solid bitorus $B \subset M$ containing a neighborhood of $N_0 \cup N_1 \cup N_3$, proving the lemma in this case.

Let us now assume that $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ contains also some leaf of \mathcal{F} . By Lemma 8.1, $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ is union of N_0 and separatrices of N_0 . Up to reordering the regions R_1 and R_2 , we have two possibilities: either $\mathcal{C}(N_1, \mathcal{F}) \cap U \subset R_1$ and $\mathcal{C}(N_2, \mathcal{F}) \cap U \subset R_3$, or else $\mathcal{C}(N_1, \mathcal{F}) \cap U \cap R_j \neq \emptyset$ for $j = 1, 2$ and $\mathcal{C}(N_2, \mathcal{F}) \cap U \subset R_3$.

Case 1. Assume $\mathcal{C}(N_1, \mathcal{F}) \cap U \subset R_1$ and $\mathcal{C}(N_2, \mathcal{F}) \cap U \subset R_3$. Then given a leaf $L_1 \in \mathcal{C}(N_1, \mathcal{F})$ we have that $L_1 \setminus (L_1 \cap U)$ is a torus minus a disc; and given a leaf $L_2 \in \mathcal{C}(N_2, \mathcal{F})$ we have that $L_2 \setminus (L_2 \cap U)$ is a torus minus a cylinder, neighborhood of a curve that bounds a disc. Notice that if $\Lambda(N_0)$ is the union of N_0 with all the separatrices accumulating to N_0 , then $\Lambda(N_0) \setminus N_0$ may have one or two connected components. Suppose it is connected. Then, since the regions R_1 and R_3 are adjacent, Theorem 6.3 implies that $L_1 \setminus (L_1 \cap U(N_0))$ and $L_2 \setminus (L_2 \cap U(N_0))$ are homeomorphic what is absurd, so this case cannot occur. Thence $\Lambda(N_0) \setminus N_0$ has two connected components $\Lambda_1 \cup \Lambda_2$, say with $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \Lambda_1 \neq \emptyset$. We have $\partial\mathcal{C}(N_1, \mathcal{F}) = \Lambda_1 \cup N_0$ and Theorem 6.3 implies that for a leaf $L_2 \in \mathcal{C}(N_2, \mathcal{F})$ we have that $L_2 \setminus U(N_0)$ has two connected components L_2^1 and L_2^2 with L_2^1 close to Λ_1 in the sense of Proposition 6.1. Therefore, L_2^1 is a torus minus a disc and the same holds for $\Lambda_1 \setminus (\Lambda_1 \cap U(N_0))$. By the local description of \mathcal{F} in $U(N_0)$ we have that Λ_1 is homeomorphic to torus pinched at a point. By Proposition 6.1, given $L_2 \in \mathcal{C}(N_2, \mathcal{F})$ the component L_2^1 is a torus minus a disc. By the local form of \mathcal{F} in $U(N_0)$ we have that $L_2 \cap U(N_0)$ is a cylinder and, since L_2 is a torus and L_2^1 is a torus minus a disc, we have that L_2^2 is a disc. A leaf L contained in the interior of the region bounded by $\Lambda(N_0)$ and close enough to Λ_2 must be a 2-sphere (it is the

union of two discs, one given by Theorem 6.3 and the homeomorphism with L_2^2 and the other given by the local type of the leaves of \mathcal{F} in the region R_2), proving the lemma in this case.

Case 2. We have $\mathcal{C}(N_1, \mathcal{F}) \cap U \cap R_j \neq \emptyset$ for $j = 1, 2$ and $\mathcal{C}(N_2, \mathcal{F}) \cap U \subset R_3$. It is then clear that all separatrices of N_0 are contained in $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$. Since there are no saddle connections we have that $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) = \Lambda(N_0)$ is the union of N_0 and all the separatrices of N_0 . Because $\mathcal{C}(N_1, \mathcal{F}) \cap R_j \neq \emptyset$ for $j = 1, 2$ we have that $\Lambda(N_0) \setminus N_0$ is connected. Also, given a leaf $L_1 \in \mathcal{C}(N_1, \mathcal{F})$ we have that $L_1 \setminus (L_1 \cap U)$ is connected and equal to a torus minus two discs. Thus $L_1 \setminus (L_1 \cap U)$ is a cylinder with a single handle. Given a leaf $L_2 \subset \mathcal{C}(N_2, \mathcal{F})$ we have that the intersection $L_2 \cap U$ is a cylinder. Theorem 6.3 implies that $L_1 \setminus (L_1 \cap U)$ and $L_2 \setminus (L_2 \cap U)$ are homeomorphic to $\Lambda(N_0) \setminus (\Lambda(N_0) \cap U)$. This and the local description of \mathcal{F} in U show that the leaves $L_2 \subset \mathcal{C}(N_2, \mathcal{F})$ are homeomorphic to the union of a torus minus two discs with a cylinder $S^1 \times [0, 1]$, so L_2 is a bitorus. On the other hand Corollary 4.10 implies that the leaves $L_2 \subset \partial\mathcal{C}(N_2, \mathcal{F})$ must be tori, hence this case cannot occur. \square

Remark 8.5. Consider the above setting with the irreducible component being of bitorus type. Let B be the invariant bitorus in that proof and let $\Omega \subset M$ be defined as the union of B and all leaves L homeomorphic to the bitorus and such that L bounds a region $R(L)$ containing $N_0 \cup N_1 \cup N_2$. By the above arguments, Ω contains a neighborhood of $N_0 \cup N_1 \cup N_2$ and all separatrices of N_0 . We claim that $\overline{\Omega} = \Omega \cup \{N'_0\}$ for another saddle N'_0 . Indeed, by the local triviality of \mathcal{F} in a neighborhood of a compact leaf, Ω is open in $M \setminus \text{sing}(\mathcal{F})$. By Corollary 4.10 every leaf in a neighborhood of a center component is a torus or a sphere, therefore $\partial\Omega$ contains no center. Thus either $\partial\Omega$ is empty or contains a (single) saddle say, N'_0 . If $\partial\Omega = \emptyset$ then $M = \Omega$ and we have $\text{sing}(\mathcal{F})$ consisting of two centers and one saddle, contradicting the hypothesis $c(\mathcal{F}) > 2s(\mathcal{F})$. Therefore we have another saddle N'_0 such that $N'_0 \subset \partial\Omega$ and thus $s(\mathcal{F}) \geq 2$. This remark will be used in the proof of Theorem 8.20.

Lemma 8.6 ($N_0 = \{q\}$, centers of mixed dimensions). *Let N_0 be an isolated saddle in pairing with N_1, N_2 , where N_1 is isolated and N_2 is non-isolated. Then either $N_0 \leftrightarrow N_1$ is a trivial pairing or it is a non-trivial pairing as in Figure 11 (left picture). In the second case we can choose an invariant region R containing N_0 and N_1 , diffeomorphic to*

a solid torus, where we can replace $\mathcal{F}|_R$ by a trivial foliation by tori around a non-isolated center.

Proof. By hypothesis N_0 and N_1 are isolated singularities and we assume the pairing $N_0 \leftrightarrow N_1$ is non-trivial. By Lemma 8.3 we have a picture as in Figure 11. We claim that the leaves L close enough to $\partial\mathcal{C}(N_1, \mathcal{F})$, but not contained in $\mathcal{C}(N_1, \mathcal{F})$, are diffeomorphic to tori. Indeed such a leaf is a compact orientable surface and there two possibilities: it is a torus obtained as the union of two cylinders, one “bigger” given by the triviality of the holonomy of $\partial\mathcal{C}(N_1, \mathcal{F})$ and other “smaller” given by the local structure of \mathcal{F} around the (isolated) saddle N_0 , or it is the union of a “big” cylinder with two discs, resulting in a 2-sphere. Nevertheless, in this last case all leaves near N_0 , except for the separatrices, are spheres and this is not possible by a standard homology argument (see Lemma 3 in [3] for details). We can therefore replace \mathcal{F} in V by a foliation with a non-isolated center as singular set. This proves the lemma. \square

Notice, moreover, that a leaf L as in the proof above necessarily bounds two solid tori invariant by \mathcal{F} : one is $R(L) \subset \mathcal{C}(N_2, \mathcal{F})$, and the other is union of (the singular solid torus) $\overline{\mathcal{C}(N_1, \mathcal{F})}$ with a small neighborhood of $\partial\mathcal{C}(N_1, \mathcal{F})$ bounded by L . Hence the union $\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}$ must be all of M , and therefore M is diffeomorphic to a Lens space: these are the only oriented 3-manifolds that are a union of two solid tori, glued along their common boundary (see [9] pages 20–21).

Summarizing the above discussion we obtain:

Proposition 8.7. *Let \mathcal{F} be a closed Bott-Morse foliation on a closed 3-manifold M satisfying either $c(\mathcal{F}) > 2s(\mathcal{F})$ or $\text{sing}(\mathcal{F})$ has pure dimension and $c(\mathcal{F}) > s(\mathcal{F})$. Suppose there is an isolated saddle singularity $N_0 \subset \text{sing}(\mathcal{F})$ in pairing with two centers $N_1, N_2 \subset \text{sing}(\mathcal{F})$. Then either N_0, N_1, N_2 belong to a same disconnected component of \mathcal{F} or one of the centers, say N_1 , must be isolated and we have:*

- (i) *If N_2 is also isolated then one of the pairings $N_0 \leftrightarrow N_j$ is trivial.*
- (ii) *If N_2 is non-isolated then there are two possibilities: either the pairing $N_0 \leftrightarrow N_1$ is trivial or there is a compact region containing $N_0 \cup N_1$, diffeomorphic to a solid torus, where \mathcal{F} can be replaced by a compact foliation in $S^1 \times \overline{D}^2$ with a one-dimensional center-type component as singular set and invariant boundary $S^1 \times S^1$.*

Notice that in case (i) we can perform a foliated surgery to eliminate an isolated center and an isolated saddle, while in case (ii), either we can eliminate an isolated center and an isolated saddle, or else we can replace an isolated center and an isolated saddle by a non-isolated center. In all cases the condition $c(\mathcal{F}) > 2s(\mathcal{F})$ is preserved, and if the singularities are all isolated and we start with $c(\mathcal{F}) > s(\mathcal{F})$, then this same condition is preserved by all the above reductions.

8.2 Non-isolated saddle

Now we study the possible pictures of a non-isolated saddle N_0 at the common boundary of two basins $\mathcal{C}(N_j, \mathcal{F}), j = 1, 2$. We first have a result about the boundary of the basin of a non-isolated center in pairing with a non-isolated saddle.

Lemma 8.8. *Assume \mathcal{F} satisfies the inequality $c(\mathcal{F}) > s(\mathcal{F})$. Let $N \subset \text{sing}(\mathcal{F})$ be a one-dimensional center-type component which is in pairing with some one-dimensional saddle type component N_0 . Then $\partial\mathcal{C}(N, \mathcal{F})$ is homeomorphic to a torus or to the union of two tori intersecting along a common circle which is a parallel.*

The pairings in Figure 10 illustrate both possibilities in this lemma.

Proof. Both N and N_0 are non-isolated. Take a distinguished neighborhood U of N_0 in M where \mathcal{F} is equivalent to the lift to a trivial bundle over $N_0 \cong S^1$ of a foliation \mathcal{F}_1 with a Morse singularity in a neighborhood $V \subset \mathbb{R}^2$ of the origin $0 \in \mathbb{R}^2$; we identify V with a disc transversal to N_0 . Notice that U is divided into four conical sectors by the separatrices of N_0 . Consider a sequence of leaves of \mathcal{F} that has N_0 in its closure, and let \mathcal{L} be one of these leaves. The trace of \mathcal{L} in the transversal V consists of i arcs, for some $i = 1, \dots, 4$. We claim i must be either 1 or 2. In fact, if $i = 4$, then the closure $\overline{\mathcal{C}(N, \mathcal{F})}$ would be also an open set in M , so $\overline{\mathcal{C}(N, \mathcal{F})} = M$, contradicting the hypothesis $c(\mathcal{F}) > s(\mathcal{F})$. If $i = 3$, then given a leaf $L \subset \mathcal{C}(N, \mathcal{F})$ we have $L \setminus (L \cap U)$ homeomorphic to a torus minus three "parallel" strips, this has three connected components. On the other hand, given any leaf $L_1 \not\subset \mathcal{C}(N, \mathcal{F})$ which is sufficiently near N_0 we have that $L_1 \cap U$ is homeomorphic to a strip and therefore $L_1 \setminus (L_1 \cap U)$ has one connected component, which is a contradiction to Theorem 6.5. Hence i is 1 or 2. Suppose that $i = 1$. A leaf $L \subset \mathcal{C}(N, \mathcal{F})$ is a torus and the intersection $L \cap U$ is a strip in L so that $L \setminus (L \cap U)$ is a cylinder: Notice that *a priori* the strip we remove

from the torus is not a neighborhood of a parallel, but a neighborhood of a curve of type $(1, p)$, up to isotopy. Nevertheless the complement of such neighborhood is also a cylinder, by Remark 8.9 below. Same observation applies to Λ below.

By Theorem 6.3 if we denote by Λ the union of N_0 and the separatrices of N_0 contained in $\partial\mathcal{C}(N, \mathcal{F})$, then $\Lambda \setminus (L \cap U)$ is a cylinder. By Proposition 5.2, $\Lambda \cap U$ is a product of S^1 by an open interval. Therefore Λ is the union of two cylinders and $\Lambda \setminus N_0$ is a smooth manifold. This shows that Λ is homeomorphic to a torus. Since \mathcal{F} has no saddle connections we have $\partial\mathcal{C}(N, \mathcal{F}) = \Lambda$. Finally, if $i = 2$ using Theorem 6.3 and Proposition 5.2 and reasoning as above we conclude that $\partial\mathcal{C}(N, \mathcal{F})$ consists of two tori that meet at N_0 (see Figure 8). \square

Remark 8.9. Let γ be a $(1, p)$ type closed curve in the torus $T^2 \cong S^1 \times S^1$. Then there is a homeomorphism of the torus mapping γ onto a $(1, 0)$ type curve (to see this take any matrix A of determinant one mapping $(1, p)$ onto $(1, 0)$ and consider the corresponding torus diffeomorphism). Hence if we consider a tubular neighborhood of γ in the torus, then its complement is homeomorphic to a cylinder. In particular a closed oriented surface obtained as the gluing of two such complements is diffeomorphic to the torus*.

Now we have:

Lemma 8.10 ($N_0 \cong S^1$, isolated centers, non-singular intersection of boundaries). *Suppose that N_1 and N_2 are isolated centers and $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) \not\subset \text{sing}(\mathcal{F})$. Then $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ is a 2-disc bounded by N_0 . Moreover, $\mathcal{C}(N_1, \mathcal{F}) \cup \mathcal{C}(N_2, \mathcal{F})$ is a closed 3-ball and the pairing $N_0 \leftrightarrow (N_1, N_2)$ is trivial.*

Proof. Since N_1 and N_2 are isolated centers, $\mathcal{C}(N_j, \mathcal{F})$, $j = 1, 2$, are open balls where the foliation is by concentric spheres S^2 . We denote by $\Lambda(N_0)$ the union of N_0 with the separatrices through N_0 . By Theorem 3.2 the holonomy of $N_0 \cong S^1$ is trivial and therefore \mathcal{F} has a bundle-type structure in a neighborhood $U \subset M$ of N_0 . The boundary $\partial\mathcal{C}(N_j, \mathcal{F})$ is a subvariety which is a limit of spheres S^2 pinched along $N_0 \cong S^1$ with a bundle-type structure in U . Thus $\partial\mathcal{C}(N_j, \mathcal{F})$ is a union of two 2-discs $D_j^{(1)}$ and $D_j^{(2)}$ along their common boundary S^1 . Since by hypothesis $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ contains some leaf of \mathcal{F} , not only N_0 ,

*We are grateful to A. Verjovsky for this remark.

it follows from the local picture of \mathcal{F} in U that, up to reordering we have $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) = D_1^{(2)} = D_2^{(1)}$ and the union $\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}$ is homeomorphic to the closed three-ball B^3 .

In fact we can assume that $\partial(\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}) = D_1^{(1)} \cup D_2^{(2)} \cup N_0$, so we conclude that there are neighborhoods W_ν of $\partial(\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}) = D_1^{(1)} \cup D_2^{(2)} \cup N_0$ such that $\lim W_\nu = \partial(\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}) = D_1^{(1)} \cup D_2^{(2)} \cup N_0$. If a non-separatrix leaf $L \in \mathcal{F}$ intersects W_ν and L is not contained in $\partial(\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})})$, then $(L \cap W_\nu)$ is either a single disc or the union of two discs, by Theorem 6.3. Moreover, the intersection $L \cap U$ is diffeomorphic to a bundle over S^1 with fiber the interval $(-\epsilon, \epsilon)$. This holds for leaves L nearby $D_1^{(1)}$ and leaves L nearby $D_2^{(2)}$, so we have:

Claim 8.11. *We can choose a compact neighborhood $V \subset M$ of $\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}$ diffeomorphic to a solid cylinder $D^2 \times [0, 1]$ with boundary ∂V consisting of two invariant discs $D_1 \cong D^2 \times \{0\}$ and $D_2 = D^2 \times \{1\}$ and a transverse open cylinder $\Sigma \cong S^1 \times (0, 1)$. The intersection $\text{sing}(\mathcal{F}) \cap V$ consists of N_0, N_1, N_2 and no other component of $\text{sing}(\mathcal{F})$.*

Using this claim we can replace $\mathcal{F}|_V$ by a trivial foliation by discs. \square

Lemma 8.12 ($N_0 \cong S^1$, isolated centers, singular intersection of boundaries). *Suppose that N_1 and N_2 are isolated and $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) = N_0$. Then there is an invariant region R containing N_0, N_1, N_2 , diffeomorphic to $S^2 \times [0, 1]$, and $\mathcal{F}|_R$ can be replaced by a regular foliation by 2-spheres, therefore we can eliminate the three components N_0, N_1, N_2 at once.*

Proof. The situation is depicted in Figure 18 below.

A leaf L nearby but not contained in $\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}$ is a 2-sphere: indeed, the intersection of this leaf with a suitable product type neighborhood U , where \mathcal{F} is of product type, is a strip. On the other hand, $\partial\mathcal{C}(N_j, \mathcal{F}) \setminus \partial(\mathcal{C}(N_j, \mathcal{F}) \cap U)$ is a 2-sphere minus a strip around the equator so that $\partial\mathcal{C}(N_j, \mathcal{F}) \setminus \partial(\mathcal{C}(N_j, \mathcal{F}) \cap U)$ consists of two disjoint 2-discs. Thus, as in the lemmas above, Theorem 6.3 implies that $L \setminus (L \cap U)$ is homeomorphic to the disjoint union of two 2-discs. Hence, L is the union of two discs D^2 and one strip $[-1, 1] \times S^1$. Since it is compact and orientable, the only possibility is L diffeomorphic to S^2 . Therefore we have a compact invariant neighborhood R of $\overline{\mathcal{C}(N_1, \mathcal{F})} \cup \overline{\mathcal{C}(N_2, \mathcal{F})}$ diffeomorphic to $S^2 \times [0, 1]$ and the lemma follows. \square

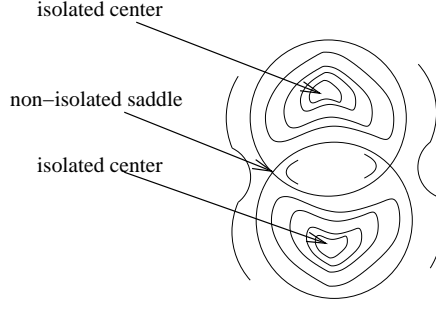


Figure 18:

Lemma 8.13 ($N_0 \cong S^1$, non-isolated centers, singular intersection of boundaries). *Assume now that $N_0 \subset \text{Sad}(\mathcal{F})$ and $N_1, N_2 \subset \text{Cent}(\mathcal{F})$ are all of dimension one. If $N_0 = \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$, then there is an invariant solid torus V containing N_0, N_1, N_2 . Therefore, we can replace $\mathcal{F}|_V$ by a foliation by concentric tori.*

Proof. By hypothesis N_0, N_1, N_2 have dimension one. Given $N_0 \subset \text{sing}(\mathcal{F})$ we denote by $\Lambda(N_0)$ the union of N_0 with the separatrices through N_0 . By Proposition 5.2 \mathcal{F} has a product structure in a distinguished neighborhood $U \subset M$ of N_0 diffeomorphic to a product $D \times S^1$ where $D \subset \mathbb{R}^2$ is a product of open intervals. The separatrix $\Lambda(N_0)$ divides U into four regions R_1, R_2, R_3, R_4 and we can assume that R_j is adjacent to R_{j+1} and R_4 to R_1 . Considering the several possibilities for the trace of $\mathcal{C}(N_1, \mathcal{F})$ and $\mathcal{C}(N_2, \mathcal{F})$ in U under the assumption that $\mathcal{C}(N_1, \mathcal{F}) \cap \mathcal{C}(N_2, \mathcal{F}) = N_0$ we obtain, after reordering the regions R_j , the only possibility is: $\mathcal{C}(N_1, \mathcal{F}) \cap U \subset R_1$ and $\mathcal{C}(N_2, \mathcal{F}) \cap U \subset R_3$.

Claim 8.14. *We can choose a solid torus $V \subset M$ with boundary ∂V invariant by \mathcal{F} and such that V contains a neighborhood of N_0 , V also contains $\Lambda(N_0)$ and $\text{sing}(\mathcal{F}) \cap V$ consists of N_0 and the two other components N_1, N_2 of center-type.*

Let us prove this claim: given a leaf $L_j \subset \mathcal{C}(N_j, \mathcal{F})$ of \mathcal{F} that intersects U we have that $L_j \setminus (L_j \cap U)$ is a torus minus a strip, *i.e.*, a cylinder. Thus, by Proposition 6.1, given an outer leaf L such that $L \cap \mathcal{C}(N_j, \mathcal{F}) = \emptyset$ one has that L is obtained as the union of two cylinders glued by their boundaries, thus L is a torus. The basins $\mathcal{C}(N_j, \mathcal{F})$ are solid tori. Proposition 8.8 then shows that L bounds a region diffeomorphic to the solid torus, obtained by the union of two solid cylinders

glued by their common boundaries. This region is invariant and contains the singularities N_0, N_1, N_2 , proving the claim and completing the proof of (8.13). \square

Lemma 8.15 ($N_0 \cong S^1$, non-isolated centers, nonsingular intersection of boundaries). *Suppose that $c(\mathcal{F}) > s(\mathcal{F})$. If $N_0 \subset \text{Sad}(\mathcal{F})$ and $N_1, N_2 \subset \text{Cent}(\mathcal{F})$ are all of dimension one and $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) \not\subset \text{sing}(\mathcal{F})$ then there is an invariant solid torus V containing N_0, N_1, N_2 and we can replace $\mathcal{F}|_V$ by a foliation by concentric tori.*

Proof. We use the same notation above. Considering the several possibilities for the trace of $\mathcal{C}(N_1, \mathcal{F})$ and $\mathcal{C}(N_2, \mathcal{F})$ in U under the assumption that $\mathcal{C}(N_1, \mathcal{F}) \cap \mathcal{C}(N_2, \mathcal{F}) = N_0$ we obtain, after reordering the regions R_j , that the only possibilities are:

First case. $\mathcal{C}(N_1, \mathcal{F}) \cup \mathcal{C}(N_2, \mathcal{F}) \cup \Lambda(N_0)$ is a neighborhood of N_0 .

This corresponds to the case where four regions R_j are intersected by the basins and the region $\Omega \subset M$ obtained as the union of these basins and $\Lambda(N_0)$ is open and closed in M . Thus $\Omega = M$ and $\text{sing}(\mathcal{F}) = N_0 \cup N_1 \cup N_2$, contradicting our hypothesis that $c(\mathcal{F}) > s(\mathcal{F})$. This case cannot occur.

Second case. In this case $\mathcal{C}(N_1, \mathcal{F}) \cap U \subset R_1$ and $\mathcal{C}(N_2, \mathcal{F}) \cap U \subset R_2 \cup R_4$ with $\mathcal{C}(N_2, \mathcal{F}) \cap R_2 \neq \emptyset$ and $\mathcal{C}(N_2, \mathcal{F}) \cap R_4 \neq \emptyset$.

Claim 8.16. *We can choose an invariant solid torus $V \subset M$ with boundary ∂V , invariant by \mathcal{F} and such that V contains a neighborhood of N_0 ; also, V contains $\Lambda(N_0)$ and $\text{sing}(\mathcal{F}) \cap V$ consists of N_0 and two other components N_1, N_2 of center-type.*

Proof of the claim. Using the same notation as above we consider L an outer leaf of \mathcal{F} . Then $L \cap U$ has two connected components (those close to $\mathcal{C}(N_2, \mathcal{F}) \cap U$) so that by Proposition 6.1, L is the union of two strips (the result of deleting two parallel strips in the torus) glued by their boundaries, thus resulting in a torus. This leaf bounds a region which is the union of two solid cylinders glued by their common boundaries, resulting in a solid torus. \square

This completes the proof of the lemma. \square

The remaining cases are the content of the following lemmas:

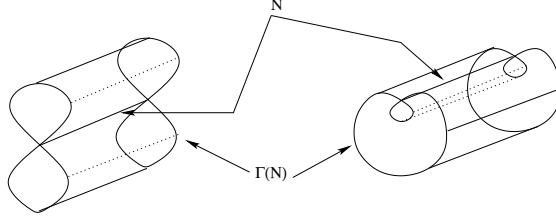


Figure 19:

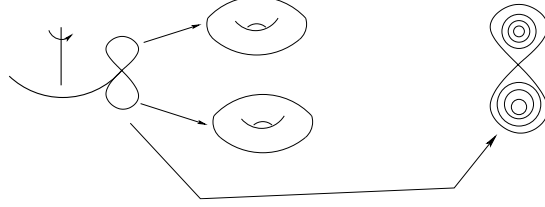


Figure 20:

Lemma 8.17 ($N_0 \cong S^1$, mixed dimensions, singular intersection of boundaries). *Assume now that $N_0 \subset \text{sing}(\mathcal{F})$ is a non-isolated saddle component such that $N_0 = \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ for center-type components $N_1, N_2 \subset \text{sing}(\mathcal{F})$, where N_1 is non-isolated and N_2 is isolated. Then there is a closed invariant ball $B^3 \subset M$ containing $N_0 \cup N_1 \cup N_2$, such that we can replace \mathcal{F} in B^3 by a foliation with an isolated center.*

Proof. We keep the notation of the proof of the above lemmas. We have the following possibilities, up to reordering the regions R_1, \dots, R_4 defined by $\Lambda(N_0)$ in the distinguished neighborhood U .

First case. $\mathcal{C}(N_2, \mathcal{F}) \cap U \subset R_1$ and $\partial\mathcal{C}(N_1, \mathcal{F}) \cap U \subset R_3$. A leaf $L_2 \subset \mathcal{C}(N_2, \mathcal{F})$ is a 2-sphere and $L_2 \cap U$ is of neighborhood of an equator in L_2 , so that $L_2 \setminus (L_2 \cap U)$ is the disjoint union of two discs D^+ and D^- . A leaf $L_1 \subset \mathcal{C}(N_1, \mathcal{F})$ is a torus and $L_1 \cap U$ is a strip in the torus, so that $L_1 \setminus (L_1 \cap U)$ is a cylinder. By Proposition 6.1 an exterior leaf L to $\Lambda(N_0)$ is homeomorphic to the union of the discs D^+ and D^- and a cylinder, through their common boundaries; so L is homeomorphic to the 2-sphere. Also, L bounds a region Ω containing the union $\mathcal{C}(N_1, \mathcal{F}) \cup \mathcal{C}(N_2, \mathcal{F}) \cup U$, this region is homeomorphic to the union of a solid cylinder (obtained from the solid torus $\mathcal{C}(N_1, \mathcal{F})$ by deleting the neighborhood U

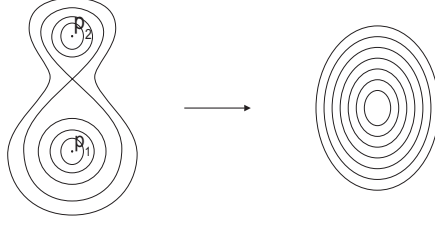


Figure 21:

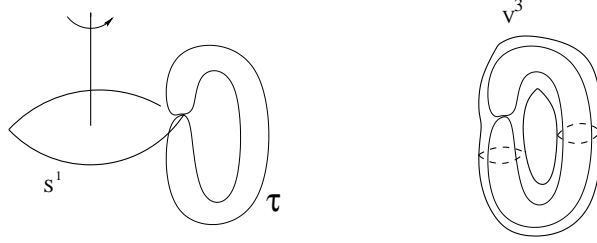


Figure 22:

which is of product type by S^1) with two solid hemispheres H^+ and H^- (obtained from the solid ball $\mathcal{C}(N_2, \mathcal{F})$ by deleting the solid intersection with U) so that Ω is a solid ball. The situation is depicted in (Figure 14).

Second case. $\mathcal{C}(N_1, \mathcal{F})$ and $\mathcal{C}(N_2, \mathcal{F})$ intersect some adjacent regions. In this case the intersection $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$ contains some separatrix of N_0 , what is not possible by hypothesis, and the lemma is proved. \square

Lemma 8.18 ($N_0 \cong S^1$, mixed dimensions, non-singular intersection of boundaries). *Assume now that $N_0 \subset \text{sing}(\mathcal{F})$ is a non-isolated saddle component such that $N_0 = \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) \not\subset \text{sing}(\mathcal{F})$ for center-type components $N_1, N_2 \subset \text{sing}(\mathcal{F})$, where N_1 is non-isolated and N_2 is isolated. Then there is a closed invariant region $\Omega \subset M$ diffeomorphic to the product $S^2 \times [0, 1]$ or to the product $S^1 \times S^1 \times [0, 1]$ containing $N_0 \cup N_1$, such that we can replace \mathcal{F} in Ω by a regular foliation by 2-spheres or tori respectively. The center N_2 appears at infinity with respect to Ω .*

Proof. We keep the notation of the proof of the above lemmas. Since by hypothesis we have $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) \not\subset \text{sing}(\mathcal{F})$, we have the

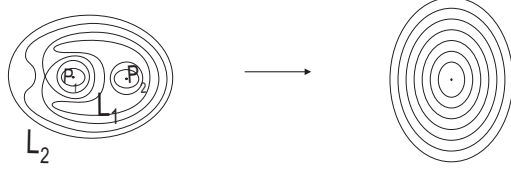


Figure 23: Transverse section showing the elimination procedure

following possibilities, up to reordering the regions R_1, \dots, R_4 defined by $\Lambda(N_0)$ in the distinguished neighborhood U .

First case. $\mathcal{C}(N_2, \mathcal{F}) \cap U \subset (R_2 \cup R_4)$, $\mathcal{C}(N_2, \mathcal{F})$ intersects R_2 and R_4 , $\partial C(N_1, \mathcal{F}) \cap U \subset R_3$ and $\mathcal{C}(N_1, \mathcal{F}) \cap R_1 = \emptyset$. Again as in the preceding lemmas, a leaf $L_2 \subset \mathcal{C}(N_2, \mathcal{F})$ is a 2-sphere and therefore its trace in U consists of two parallel strips, *i.e.*, two neighborhoods of meridians, and $L_2 \setminus (L_2 \cap U)$ is the union of two disjoint discs. A leaf $L_1 \subset \mathcal{C}(N_1, \mathcal{F})$ is a torus and its trace in U is a strip, so that $L_1 \setminus (L_1 \cap U)$ is a cylinder. A leaf L intersecting R_1 cannot belong to $\mathcal{C}(N_1, \mathcal{F})$ nor to $\mathcal{C}(N_2, \mathcal{F})$. Nevertheless the above remarks and Proposition 6.1 imply that L is also a 2-sphere. Thus we can find a region Ω bounded by leaves in $\mathcal{C}(N_2, \mathcal{F})$ diffeomorphic to S^2 , and by leaves intersecting R_1 , also diffeomorphic to S^2 . The region Ω contains N_0 and N_1 and is diffeomorphic to the product $S^2 \times [0, 1]$.

Second case. $\mathcal{C}(N_1, \mathcal{F}) \cap U \subset (R_2 \cup R_4)$, $\mathcal{C}(N_1, \mathcal{F})$ intersects R_2 and R_4 , $\partial C(N_2, \mathcal{F}) \cap U \subset R_3$ and $\mathcal{C}(N_2, \mathcal{F}) \cap R_1 = \emptyset$. In this case we apply Proposition 6.1 to compare the leaves in $\mathcal{C}(N_1, \mathcal{F})$ and the leaves in $\mathcal{C}(N_2, \mathcal{F})$ also through their traces in U . This implies that a torus minus two neighborhoods of parallel meridians is homeomorphic to a 2-sphere minus a neighborhood of the equator, what is a contradiction. Hence this case cannot happen.

Third case. $\mathcal{C}(N_1, \mathcal{F}) \cap U \subset (R_2 \cup R_4)$, $\mathcal{C}(N_1, \mathcal{F})$ intersects R_2 and R_4 , $\partial C(N_2, \mathcal{F}) \cap U \subset R_1$ and $\mathcal{C}(N_2, \mathcal{F}) \cap R_3 = \emptyset$. In this case we apply Proposition 6.1 to, via comparison with the leaves in $\mathcal{C}(N_1, \mathcal{F}) \setminus (\mathcal{C}(N_1, \mathcal{F}) \cap U)$ and the leaves in $\mathcal{C}(N_2, \mathcal{F}) \setminus (\mathcal{C}(N_2, \mathcal{F}) \cap U)$ also through their traces in U . We conclude that the leaves intersecting R_3 are tori and as in the First case we obtain an invariant region $\Omega \subset M$ diffeomorphic to $S^1 \times S^1 \times [0, 1]$, containing N_1 and N_0 where we can replace the foliation \mathcal{F} by a non-singular foliation by tori. The center N_2 is at infinity with

respect to Ω .

□

From the above lemmas we have:

Proposition 8.19. *Let \mathcal{F} be a closed Bott-Morse foliation on a closed 3-manifold M such that $c(\mathcal{F}) > 2s(\mathcal{F})$ or else $\text{sing}(\mathcal{F})$ has pure dimension and $c(\mathcal{F}) > s(\mathcal{F})$. Suppose that there is a non-isolated saddle $N_0 \subset \text{sing}(\mathcal{F})$ in pairing with two centers $N_1, N_2 \subset \text{sing}(\mathcal{F})$. Then we have the following possibilities:*

- (i) *If N_1 and N_2 are isolated then the pairing $N_0 \leftrightarrow (N_1, N_2)$ is trivial and we have the following possibilities:*
 - (i.1) *$\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) \not\subset \text{sing}(\mathcal{F})$ and there is a solid cylinder $V \simeq D^2 \times [0, 1]$ with boundary ∂V consisting of two invariant discs $D_1 \cong D^2 \times \{0\}$ and $D_2 \cong D^2 \times \{1\}$ and a transverse open cylinder $\Sigma \cong S^1 \times (0, 1)$. We have $\text{sing}(\mathcal{F}) \cap V = N_0 \cup N_1 \cup N_2$ and $\mathcal{F}|_V$ can be replaced by a trivial foliation by discs.*
 - (i.2) *If $\partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F}) = N_0$ then there is an invariant region R diffeomorphic to $S^2 \times [0, 1]$ and containing N_0, N_1, N_2 , so that we can replace \mathcal{F} in R by a regular foliation by 2-spheres.*
- (ii) *If N_1 and N_2 are non-isolated, then there is an invariant solid torus V containing N_0, N_1, N_2 and we can replace $\mathcal{F}|_V$ by a foliation by concentric tori having a single non-isolated center as singular set.*
- (iii) *If N_1 is non-isolated and N_2 is isolated, then there is a closed invariant ball $B^3 \subset M$ containing $N_0 \cup N_1 \cup N_2$, such that we can replace \mathcal{F} in B^3 by a foliation with an isolated center.*

In order to pave the way for Theorem 9.1 we compile the information in Propositions 8.7 and 8.19 in an omnibus theorem as follows:

Theorem 8.20. *Let \mathcal{F} be a closed Bott-Morse foliation on a closed oriented three-manifold M such that either $c(\mathcal{F}) > 2s(\mathcal{F})$ or else $\text{sing}(\mathcal{F})$ has pure dimension and $c(\mathcal{F}) > s(\mathcal{F})$. Suppose we have two different center components $N_1, N_2 \in \text{Cent}(\mathcal{F})$ and a saddle component $N_0 \in \text{Sad}(\mathcal{F})$ such that $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$. We have the following possibilities:*

- (1) N_0 is isolated. Then either N_0, N_1, N_2 belong to a same disconnected irreducible component of \mathcal{F} or one of the centers, say N_1 , must be isolated and we have:
- (1.i) If N_2 is also isolated then one of the pairings $N_0 \leftrightarrow N_j$ is trivial.
 - (1.ii) If N_2 is non-isolated then there are two possibilities: either the pairing $N_0 \leftrightarrow N_1$ is trivial or \mathcal{F} can be modified in a solid torus replacing $N_0 \cup N_1$ by a one-dimensional center.
 - (\star) If N_0, N_1, N_2 belong to a same irreducible component then there is an invariant compact region $\Omega \subset M$ containing these singularities and the restriction $\tilde{\mathcal{F}}$ of \mathcal{F} to $\tilde{M} = M \setminus \Omega$ is closed Bott-Morse which satisfies $c(\tilde{\mathcal{F}}) > 2s(\tilde{\mathcal{F}}) \geq 2$. Also given a center $N \subset \text{sing}(\mathcal{F}) \setminus (N_1 \cup N_2)$ we have $\mathcal{C}(N, \mathcal{F}) \cap \Omega = \emptyset$.
- (2) N_0 is non-isolated:
- (2.i) If N_1 and N_2 are isolated then the pairing $N_0 \leftrightarrow (N_1, N_2)$ is trivial.
 - (2.ii) If N_1 and N_2 are non-isolated, then we can modify \mathcal{F} in an invariant solid torus replacing N_0, N_1, N_2 by a single non-isolated center.
 - (2.iii) If N_1 is non-isolated and N_2 is isolated, then we can modify \mathcal{F} in an invariant solid ball replacing $N_0 \cup N_1 \cup N_2$ by an isolated center.

Proof. Part (2) follows from Proposition 8.19. Parts (1.i), (1.ii) follow from Proposition 8.7 while (\star) follows from this same proposition and Remark 8.5 and the following argumentation: Suppose that N_0, N_1, N_2 are in a same irreducible component $\Omega \subset M$. Since $c(\mathcal{F}) > 2s(\mathcal{F})$ there is another center N_3 distinct from N_1, N_2 and such that $\partial\mathcal{C}(N_3, \mathcal{F}) \cap N_0 = \emptyset$. If $\partial\mathcal{C}(N_3, \mathcal{F}) = \emptyset$ then by Theorem 4.8 (ii) we have $s(\mathcal{F}) = 0$, absurd. Thus by Theorem 4.8 (iii) we have $\partial\mathcal{C}(N_3, \mathcal{F}) = N'_0$ for a saddle $N'_0 \neq N_0$. This shows that $s(\mathcal{F}) \geq 2$. Thus we can consider the foliation \mathcal{F} restricted to the manifold with boundary $\tilde{M} := M \setminus \Omega$. This is a Bott-Morse foliation $\tilde{\mathcal{F}}$ with $\text{sing}(\tilde{\mathcal{F}}) = \text{sing}(\mathcal{F}) \setminus N_1 \cup N_2 \cup N_0$, so that it satisfies the inequality $c(\tilde{\mathcal{F}}) > 2s(\tilde{\mathcal{F}})$, and $s(\tilde{\mathcal{F}}) \geq 1$. \square

9 A classification theorem in dimension three

We now prove:

Theorem 9.1. *Let M^3 be a closed oriented connected 3-manifold equipped with a closed Bott-Morse foliation \mathcal{F} satisfying either $c(\mathcal{F}) > 2s(\mathcal{F})$ or else $c(\mathcal{F}) > s(\mathcal{F})$ and its singular set $\text{sing}(\mathcal{F})$ is pure-dimensional. Then:*

- (i) *there is a deformation of \mathcal{F} via foliated surgery into a compact Bott-Morse foliation on M , and the holonomy pseudogroup of the foliation is preserved off the singular set; and*
- (ii) *M is homeomorphic to the 3-sphere, a Lens space or $S^1 \times S^2$.*

Part (ii) of Theorem 9.1 is an immediate consequence of part (i) and Theorem D in [19]. Examples in Section 2.1 show that the hypothesis that either $c(\mathcal{F}) > 2s(\mathcal{F})$ or $c(\mathcal{F}) > s(\mathcal{F})$ and the singular set has pure dimension, is necessary. If $\text{sing}(\mathcal{F})$ has dimension 0 then (i) is proved in [3]; in this case M is actually the 3-sphere.

To prove this theorem we need the following elimination result:

Proposition 9.2. *Let \mathcal{F} be a closed Bott-Morse foliation on a connected closed 3-manifold M . Suppose that $c(\mathcal{F}) > 2s(\mathcal{F})$. Then either $s(\mathcal{F}) = 0$ or \mathcal{F} admits a modification into a closed Bott-Morse foliation \mathcal{F}_1 on M such that either $c(\mathcal{F}_1) = c(\mathcal{F}) - 1$ and $s(\mathcal{F}_1) = s(\mathcal{F}) - 1$ or $c(\mathcal{F}_1) = c(\mathcal{F}) - 2$ and $s(\mathcal{F}_1) = s(\mathcal{F}) - 1$.*

Proof. We fix a component $N \in \text{Cent}(\mathcal{F})$ and consider $\mathcal{C}(N, \mathcal{F})$ as usual. If $\partial\mathcal{C}(N, \mathcal{F}) = \emptyset$ then we apply Theorem 4.8 (ii) to conclude that $\text{Sad}(\mathcal{F}) = \emptyset$ and therefore $s(\mathcal{F}) = 0$. Assume therefore that $\partial\mathcal{C}(N, \mathcal{F}) \neq \emptyset$. Then by Theorem 4.8 (iii) we have $\partial\mathcal{C}(N, \mathcal{F}) \subset \text{Sad}(\mathcal{F})$ and there is some component $S(N) \in \text{Sad}(\mathcal{F}) \cap \partial\mathcal{C}(N, \mathcal{F})$. This way we define a map $\Theta: N \mapsto S(N)$ from the set of center-type components $\text{Cent}(\mathcal{F})$ into the set of saddle-type components $\text{Sad}(\mathcal{F})$. This map cannot be injective since we have the inequality $c(\mathcal{F}) > 2s(\mathcal{F})$. Therefore there are two different center components $N_1, N_2 \in \text{Cent}(\mathcal{F})$ and a saddle component $N_0 \in \text{Sad}(\mathcal{F})$ such that $N_0 \subset \partial\mathcal{C}(N_1, \mathcal{F}) \cap \partial\mathcal{C}(N_2, \mathcal{F})$. If N_0 is non-isolated the conclusion now follows from Theorem 8.20 part (2). Assume now that N_0 is isolated. According to Theorem 8.20 part (1) the modification exists if N_0, N_1, N_2 are not in a same irreducible component of \mathcal{F} . Suppose therefore that N_0, N_1, N_2 are in a same irreducible component $\Omega \subset M$. According to Theorem 8.20 (\star)

if N_0, N_1, N_2 belong to a same irreducible component then there is an invariant compact region $\Omega \subset M$ containing these singularities and the restriction $\tilde{\mathcal{F}}$ of \mathcal{F} to $\tilde{M} = M \setminus \Omega$ is closed Bott-Morse which satisfies $c(\tilde{\mathcal{F}}) > 2s(\tilde{\mathcal{F}}) \geq 2$ and such that $\mathcal{C}(N, \mathcal{F}) \cap \Omega = \emptyset$ for every center $N \subset \text{sing}(\mathcal{F})$ $N \neq N_j, j = 1, 2$. Thus, arguing as above we can find two centers \tilde{N}_1, \tilde{N}_2 and a saddle \tilde{N}_0 such that $\partial\mathcal{C}(N_1, \tilde{\mathcal{F}}) \cap \tilde{\mathcal{C}}(N_2, \tilde{\mathcal{F}}) = \tilde{N}_0$. If \tilde{N}_0 is not isolated or $\tilde{N}_0, \tilde{N}_1, \tilde{N}_2$ do not belong to the same irreducible component of $\tilde{\mathcal{F}}$ on \tilde{M} and we can obtain a closed Bott-Morse foliation $\tilde{\tilde{\mathcal{F}}}$ which is a modification of $\tilde{\mathcal{F}}$ on \tilde{M} and satisfies $c(\tilde{\tilde{\mathcal{F}}}) > 2s(\tilde{\tilde{\mathcal{F}}})$. This gives a modification of \mathcal{F} on M with the desired properties. The only remaining case is when $\tilde{N}_0, \tilde{N}_1, \tilde{N}_2$ belong to a same irreducible component $\tilde{\Omega}$ of $\tilde{\mathcal{F}}$ on \tilde{M} . Notice that this process can be repeated until we can assure that for the resulting foliation the two centers and the saddle are not in a same irreducible component. Indeed, this is the case because for any irreducible component we have two centers and one saddle. If all singularities are in irreducible components then we have $c(\mathcal{F}) = 2s(\mathcal{F})$, contradiction. A standard Finite Induction argument then ends the proof of the theorem. \square

Proof of Theorem 9.1. We proceed by induction on the number $s(\mathcal{F})$ of saddle components in $\text{sing}(\mathcal{F})$. We first assume that $s(\mathcal{F}) = 0$. Then \mathcal{F} is a compact Bott-Morse foliation with nonempty singular set and Theorem 9.1 follows from Theorem D in [19]. Now we assume that the result is valid for closed Bott-Morse foliations \mathcal{F}_1 on closed 3-manifolds satisfying $c(\mathcal{F}_1) > 2s(\mathcal{F}_1)$ and having number of saddle components not greater than $k \geq 1$, i.e., $s(\mathcal{F}_1) \leq k$. Let \mathcal{F} be a closed Bott-Morse foliation on M such that $c(\mathcal{F}) > 2s(\mathcal{F})$ and such that $s(\mathcal{F}) = k + 1$. Since $c(\mathcal{F}) > 2s(\mathcal{F})$ and $s(\mathcal{F}) \geq 1$ it follows from Proposition 9.2 that M admits a closed Bott-Morse foliation \mathcal{F}_1 for which we have $c(\mathcal{F}_1) > 2s(\mathcal{F}_1)$ and $s(\mathcal{F}_1) < s(\mathcal{F})$. This proves the theorem. \square

Proof of Theorem 9.1. This result is proved in [3] in case all components of the singular set have dimension zero. Thus we only have to prove the case where the components of the singular set have dimension one. For this it is enough to observe that if we have a pairing $N_0 \leftrightarrow (N_1, N_2)$ where all the components are of dimension one then we can always eliminate some pair of components $N_0 \leftrightarrow N_j$ and remain with a center component. Therefore, in case $\text{sing}(\mathcal{F})$ is a union of dimension one components, it is enough to have $c(\mathcal{F}) > s(\mathcal{F})$ to get the same conclusions as above.

□

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